



Linear Algebra

Lecture 6

Change of Basis

Section 7.3 in the textbook

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Motivations

- 了解向量（一階張量）與二階張量的基底轉換，有助於觀測資料的分析。
- 了解二階張量的基底轉換，可幫助了解在古典力學中的轉動慣量，電漿物理中的壓力張量，量子力學中自旋量與各種轉動量，以及物理數學中的本徵向量(eigen vectors)的意義。

Review Vectors' Expression

- Expressions of vector

- Hand writing: \vec{v} or \underline{v} or \mathbf{lv} or \vec{v}^T or \underline{v}^T or $\langle \mathbf{v} |$

- Typeface: boldface for vector, i.e., \mathbf{v} or \mathbf{v}^T

- Matrix expression of vectors: $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ or $[v_1 \quad v_2 \quad v_3]$

where the vector is defined by

$$\vec{v} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3 = \sum v_i \hat{e}_i = v_i \hat{e}_i$$

$$\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3 = \sum_i v_i \mathbf{e}_i = v_i \mathbf{e}_i$$

Matrix Expression of Vector Products

- inner product:** $\mathbf{v} \cdot \mathbf{w} = [v_1 \ v_2 \ v_3] \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

The multiplying of the projection of one vector to the other vector
- cross product:** $\mathbf{v} \times \mathbf{w} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

Size and normal direction of the area determined by the two vectors
- dyad product:** $\mathbf{vw} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} [w_1 \ w_2 \ w_3] = \begin{bmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \\ v_3 w_1 & v_3 w_2 & v_3 w_3 \end{bmatrix}$

Review of Matrix Multiplication: $U\mathbf{x}$, $\mathbf{x}^T U^T$

$$U\mathbf{x} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} \uparrow \\ \mathbf{u}_1 \\ \downarrow \end{bmatrix} + x_2 \begin{bmatrix} \uparrow \\ \mathbf{u}_2 \\ \downarrow \end{bmatrix} + x_3 \begin{bmatrix} \uparrow \\ \mathbf{u}_3 \\ \downarrow \end{bmatrix}$$

$$\mathbf{x}^T U^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{u}_1^T & \rightarrow \\ \leftarrow & \mathbf{u}_2^T & \rightarrow \\ \leftarrow & \mathbf{u}_3^T & \rightarrow \end{bmatrix}$$

$$= x_1 \begin{bmatrix} \leftarrow & \mathbf{u}_1^T & \rightarrow \end{bmatrix} + x_2 \begin{bmatrix} \leftarrow & \mathbf{u}_2^T & \rightarrow \end{bmatrix} + x_3 \begin{bmatrix} \leftarrow & \mathbf{u}_3^T & \rightarrow \end{bmatrix}$$

Review of Matrix Multiplication: UDU^T

$$\begin{aligned}
 UDU^T &= \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{u}_1^T & \rightarrow \\ \leftarrow & \mathbf{u}_2^T & \rightarrow \\ \leftarrow & \mathbf{u}_3^T & \rightarrow \end{bmatrix} \\
 &= \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ d_1\mathbf{u}_1 & d_2\mathbf{u}_2 & d_3\mathbf{u}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{u}_1^T & \rightarrow \\ \leftarrow & \mathbf{u}_2^T & \rightarrow \\ \leftarrow & \mathbf{u}_3^T & \rightarrow \end{bmatrix} \\
 &= d_1 \begin{bmatrix} \uparrow \\ \mathbf{u}_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{u}_1^T & \rightarrow \end{bmatrix} + d_2 \begin{bmatrix} \uparrow \\ \mathbf{u}_2 \\ \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{u}_2^T & \rightarrow \end{bmatrix} + d_3 \begin{bmatrix} \uparrow \\ \mathbf{u}_3 \\ \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{u}_3^T & \rightarrow \end{bmatrix}
 \end{aligned}$$

Changing Basis (Changing Coordinate System)

- \mathbf{v} in the original coordinate system:

$$\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3 = \sum_j v_j \mathbf{e}_j$$

- \mathbf{v} in the new coordinate system:

$$\mathbf{v} = v_1^* \mathbf{e}_1^* + v_2^* \mathbf{e}_2^* + v_3^* \mathbf{e}_3^* = \sum_j v_j^* \mathbf{e}_j^*$$

- How to determine v_j^* ?

Changing Basis

How to determine v_j^* ?

- 用計算式的方式（直式）證明

$$\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3 = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\mathbf{v} = v_1^* \mathbf{e}_1^* + v_2^* \mathbf{e}_2^* + v_3^* \mathbf{e}_3^* = v_1^* \begin{bmatrix} \uparrow \\ \mathbf{e}_1^* \\ \downarrow \end{bmatrix} + v_2^* \begin{bmatrix} \uparrow \\ \mathbf{e}_2^* \\ \downarrow \end{bmatrix} + v_3^* \begin{bmatrix} \uparrow \\ \mathbf{e}_3^* \\ \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \\ v_3^* \end{bmatrix}$$

Changing Basis

How to determine \mathbf{v}_j^* ?

- 用計算式的方式（直式）證明（續）

$$\mathbf{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \\ v_3^* \end{bmatrix}$$

$$\begin{bmatrix} v_1^* \\ v_2^* \\ v_3^* \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix}^{-1} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \begin{matrix} \text{=} \\ \text{=} \\ \text{=} \end{matrix} \begin{matrix} \left[\begin{array}{ccc} \leftarrow & \mathbf{e}_1^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_2^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_3^{*T} & \rightarrow \end{array} \right] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\ \text{if } \mathbf{e}_i^{*T} \mathbf{e}_j^* = \delta_{ij} \end{matrix}$$

where $\delta_{ij} = 1$, if $i = j$, but $\delta_{ij} = 0$, if $i \neq j$

Changing Basis

How to determine \mathbf{v}_j^* ?

- 用文字敘述的方式（橫式）證明，比較抽象，但為傳統證法

$$\text{Let } \mathbf{e}_j^* = \sum_i A_{ij} \mathbf{e}_i \Rightarrow \mathbf{v} = \sum_i v_i \mathbf{e}_i = \sum_j v_j^* \mathbf{e}_j^* = \sum_j v_j^* \sum_i A_{ij} \mathbf{e}_i = \sum_i \sum_j A_{ij} v_j^* \mathbf{e}_i$$

$$\Downarrow$$

$$\Rightarrow \sum_i v_i \mathbf{e}_i = \sum_i \left(\sum_j A_{ij} v_j^* \right) \mathbf{e}_i \Rightarrow v_i = \sum_j A_{ij} v_j^*$$

$$A = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = A \begin{bmatrix} v_1^* \\ v_2^* \\ v_3^* \end{bmatrix} \Rightarrow \begin{bmatrix} v_1^* \\ v_2^* \\ v_3^* \end{bmatrix} = A^{-1} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Changing Basis (Changing Coordinate System)

- **P** in the original coordinate system:

dyad product (in Physics) $\mathbf{P} = \sum_j \sum_i P_{ij} \mathbf{e}_i \mathbf{e}_j$

matrix multiplication (in Math) $P = \sum_j \sum_i P_{ij} \mathbf{e}_i \mathbf{e}_j^T$

- **P** in the new coordinate system:

$$P = \sum_j \sum_i P_{ij}^* \mathbf{e}_i^* (\mathbf{e}_j^*)^T$$

- How to determine P_{ij}^* ?

Changing Basis

How to determine P_{ij}^* ?

用文字敘述的方式
(橫式) 證明

$$\text{Let } \mathbf{e}_j^* = \sum_i A_{ij} \mathbf{e}_i \Rightarrow$$

$$A = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

$$P = \sum_j \sum_i P_{ij} \mathbf{e}_i \mathbf{e}_j^T = \sum_k \sum_l P_{lk}^* \mathbf{e}_l^* (\mathbf{e}_k^*)^T$$

$$= \sum_k \sum_l P_{lk}^* \left(\sum_i A_{il} \mathbf{e}_i \right) \left(\sum_j A_{jk} \mathbf{e}_j \right)^T$$

$$= \sum_j \sum_i \left[\sum_k \sum_l A_{il} P_{lk}^* (A^T)_{kj} \right] \mathbf{e}_i \mathbf{e}_j^T$$

$$\Rightarrow \sum_j \sum_i P_{ij} \mathbf{e}_i \mathbf{e}_j^T = \sum_j \sum_i \left[\sum_k \sum_l A_{il} P_{lk}^* (A^T)_{kj} \right] \mathbf{e}_i \mathbf{e}_j^T$$

$$\Rightarrow P_{ij} = \sum_k \sum_l A_{il} P_{lk}^* (A^T)_{kj}$$

Changing Basis

How to determine P_{ij}^* ?

用計算式的方式（直式）證明

$$\begin{aligned}
 P &= \sum_j \sum_i P_{ij} \mathbf{e}_i \mathbf{e}_j^T \\
 &= P_{11} \mathbf{e}_1 \mathbf{e}_1^T + P_{12} \mathbf{e}_1 \mathbf{e}_2^T + P_{13} \mathbf{e}_1 \mathbf{e}_3^T \\
 &\quad + P_{21} \mathbf{e}_2 \mathbf{e}_1^T + P_{22} \mathbf{e}_2 \mathbf{e}_2^T + P_{23} \mathbf{e}_2 \mathbf{e}_3^T \\
 &\quad + P_{31} \mathbf{e}_3 \mathbf{e}_1^T + P_{32} \mathbf{e}_3 \mathbf{e}_2^T + P_{33} \mathbf{e}_3 \mathbf{e}_3^T \\
 &= (P_{11} \mathbf{e}_1 + P_{21} \mathbf{e}_2 + P_{31} \mathbf{e}_3) \mathbf{e}_1^T \\
 &\quad + (P_{12} \mathbf{e}_1 + P_{22} \mathbf{e}_2 + P_{32} \mathbf{e}_3) \mathbf{e}_2^T \\
 &\quad + (P_{13} \mathbf{e}_1 + P_{23} \mathbf{e}_2 + P_{33} \mathbf{e}_3) \mathbf{e}_3^T
 \end{aligned}$$



$$\begin{aligned}
 &= (P_{11} \begin{bmatrix} \uparrow \\ \mathbf{e}_1 \\ \downarrow \end{bmatrix} + P_{21} \begin{bmatrix} \uparrow \\ \mathbf{e}_2 \\ \downarrow \end{bmatrix} + P_{31} \begin{bmatrix} \uparrow \\ \mathbf{e}_3 \\ \downarrow \end{bmatrix}) \mathbf{e}_1^T \\
 &\quad + (P_{12} \begin{bmatrix} \uparrow \\ \mathbf{e}_1 \\ \downarrow \end{bmatrix} + P_{22} \begin{bmatrix} \uparrow \\ \mathbf{e}_2 \\ \downarrow \end{bmatrix} + P_{32} \begin{bmatrix} \uparrow \\ \mathbf{e}_3 \\ \downarrow \end{bmatrix}) \mathbf{e}_2^T \\
 &\quad + (P_{13} \begin{bmatrix} \uparrow \\ \mathbf{e}_1 \\ \downarrow \end{bmatrix} + P_{23} \begin{bmatrix} \uparrow \\ \mathbf{e}_2 \\ \downarrow \end{bmatrix} + P_{33} \begin{bmatrix} \uparrow \\ \mathbf{e}_3 \\ \downarrow \end{bmatrix}) \mathbf{e}_3^T
 \end{aligned}$$

Changing Basis

How to determine P_{ij}^* ?

計算式的方式（直式）證明（續）

$$\begin{aligned}
 P = & (P_{11} \begin{bmatrix} \uparrow \\ \mathbf{e}_1 \\ \downarrow \end{bmatrix} + P_{21} \begin{bmatrix} \uparrow \\ \mathbf{e}_2 \\ \downarrow \end{bmatrix} + P_{31} \begin{bmatrix} \uparrow \\ \mathbf{e}_3 \\ \downarrow \end{bmatrix}) \mathbf{e}_1^T = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} P_{11} \\ P_{21} \\ P_{31} \end{bmatrix} \left[\leftarrow \mathbf{e}_1^T \rightarrow \right] \\
 & + (P_{12} \begin{bmatrix} \uparrow \\ \mathbf{e}_1 \\ \downarrow \end{bmatrix} + P_{22} \begin{bmatrix} \uparrow \\ \mathbf{e}_2 \\ \downarrow \end{bmatrix} + P_{32} \begin{bmatrix} \uparrow \\ \mathbf{e}_3 \\ \downarrow \end{bmatrix}) \mathbf{e}_2^T + \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} P_{12} \\ P_{22} \\ P_{32} \end{bmatrix} \left[\leftarrow \mathbf{e}_2^T \rightarrow \right] \\
 & + (P_{13} \begin{bmatrix} \uparrow \\ \mathbf{e}_1 \\ \downarrow \end{bmatrix} + P_{23} \begin{bmatrix} \uparrow \\ \mathbf{e}_2 \\ \downarrow \end{bmatrix} + P_{33} \begin{bmatrix} \uparrow \\ \mathbf{e}_3 \\ \downarrow \end{bmatrix}) \mathbf{e}_3^T + \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} P_{13} \\ P_{23} \\ P_{33} \end{bmatrix} \left[\leftarrow \mathbf{e}_3^T \rightarrow \right]
 \end{aligned}$$

Changing Basis

How to determine P_{ij}^* ?

計算式的方式（直式）證明（續）

$$P = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} P_{11} \\ P_{21} \\ P_{31} \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{e}_1^T & \rightarrow \end{bmatrix} + \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} P_{12} \\ P_{22} \\ P_{32} \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{e}_2^T & \rightarrow \end{bmatrix} + \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} P_{13} \\ P_{23} \\ P_{33} \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{e}_3^T & \rightarrow \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \left(\begin{bmatrix} P_{11} \\ P_{21} \\ P_{31} \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{e}_1^T & \rightarrow \end{bmatrix} + \begin{bmatrix} P_{12} \\ P_{22} \\ P_{32} \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{e}_2^T & \rightarrow \end{bmatrix} + \begin{bmatrix} P_{13} \\ P_{23} \\ P_{33} \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{e}_3^T & \rightarrow \end{bmatrix} \right)$$

$$= \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{e}_1^T & \rightarrow \\ \leftarrow & \mathbf{e}_2^T & \rightarrow \\ \leftarrow & \mathbf{e}_3^T & \rightarrow \end{bmatrix}$$

Changing Basis

How to determine P_{ij}^* ?

計算式的方式（直式）證明（續）

Namely,

$$P = \sum_j \sum_i P_{ij} \mathbf{e}_i \mathbf{e}_j^T = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{e}_1^T & \rightarrow \\ \leftarrow & \mathbf{e}_2^T & \rightarrow \\ \leftarrow & \mathbf{e}_3^T & \rightarrow \end{bmatrix}$$

Likewise

$$P = \sum_k \sum_l P_{lk}^* \mathbf{e}_l^* (\mathbf{e}_k^*)^T = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} P_{11}^* & P_{12}^* & P_{13}^* \\ P_{21}^* & P_{22}^* & P_{23}^* \\ P_{31}^* & P_{32}^* & P_{33}^* \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{e}_1^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_2^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_3^{*T} & \rightarrow \end{bmatrix}$$

Changing Basis

How to determine P_{ij}^* ?

計算式的方式（直式）證明（續）

$$\text{Since } \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \leftarrow & \mathbf{e}_1^T & \rightarrow \\ \leftarrow & \mathbf{e}_2^T & \rightarrow \\ \leftarrow & \mathbf{e}_3^T & \rightarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

it yields

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} P_{11}^* & P_{12}^* & P_{13}^* \\ P_{21}^* & P_{22}^* & P_{23}^* \\ P_{31}^* & P_{32}^* & P_{33}^* \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{e}_1^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_2^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_3^{*T} & \rightarrow \end{bmatrix}$$

Changing Basis

How to determine P_{ij}^* ?

計算式的方式 (直式) 證明 (續)

$$\begin{aligned}
 P &= \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} P_{11}^* & P_{12}^* & P_{13}^* \\ P_{21}^* & P_{22}^* & P_{23}^* \\ P_{31}^* & P_{32}^* & P_{33}^* \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{e}_1^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_2^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_3^{*T} & \rightarrow \end{bmatrix} \\
 \Rightarrow & \begin{bmatrix} P_{11}^* & P_{12}^* & P_{13}^* \\ P_{21}^* & P_{22}^* & P_{23}^* \\ P_{31}^* & P_{32}^* & P_{33}^* \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix}^{-1} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{e}_1^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_2^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_3^{*T} & \rightarrow \end{bmatrix}^{-1} \\
 \Rightarrow & \begin{bmatrix} P_{11}^* & P_{12}^* & P_{13}^* \\ P_{21}^* & P_{22}^* & P_{23}^* \\ P_{31}^* & P_{32}^* & P_{33}^* \end{bmatrix} = \begin{bmatrix} \leftarrow & \mathbf{e}_1^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_2^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_3^{*T} & \rightarrow \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix}
 \end{aligned}$$

if $\mathbf{e}_i^{*T} \mathbf{e}_j^* = \delta_{ij}$

Summary

When we change the basis from $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $B^* = \{\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*\}$ the representation of \mathbf{v} also changes

$$\begin{aligned}
 [\mathbf{v}]_{B^*} &= \begin{bmatrix} v_1^* \\ v_2^* \\ v_3^* \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^{*T} & \mathbf{e}_2^{*T} & \mathbf{e}_3^{*T} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}^{-1} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\
 \Rightarrow [\mathbf{v}]_{B^*} &= \begin{bmatrix} v_1^* \\ v_2^* \\ v_3^* \end{bmatrix} = \begin{bmatrix} \leftarrow & \mathbf{e}_1^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_2^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_3^{*T} & \rightarrow \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\
 \text{if } \mathbf{e}_i^{*T} \mathbf{e}_j^* &= \delta_{ij}
 \end{aligned}$$

Summary

When we change the basis from $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $B^* = \{\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*\}$ the representation of \mathbf{P} also changes

$$\begin{aligned}
 [\mathbf{P}]_{B^*} &= \begin{bmatrix} P_{11}^* & P_{12}^* & P_{13}^* \\ P_{21}^* & P_{22}^* & P_{23}^* \\ P_{31}^* & P_{32}^* & P_{33}^* \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix}^{-1} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{e}_1^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_2^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_3^{*T} & \rightarrow \end{bmatrix}^{-1} \\
 \Rightarrow [\mathbf{P}]_{B^*} &= \begin{bmatrix} P_{11}^* & P_{12}^* & P_{13}^* \\ P_{21}^* & P_{22}^* & P_{23}^* \\ P_{31}^* & P_{32}^* & P_{33}^* \end{bmatrix} = \begin{bmatrix} \leftarrow & \mathbf{e}_1^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_2^{*T} & \rightarrow \\ \leftarrow & \mathbf{e}_3^{*T} & \rightarrow \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_1^* & \mathbf{e}_2^* & \mathbf{e}_3^* \\ \downarrow & \downarrow & \downarrow \end{bmatrix}
 \end{aligned}$$

if $\mathbf{e}_i^{*T} \mathbf{e}_j^* = \delta_{ij}$