太空科學研究所

Linear Algebra
Lecture 5 (Chap. 5)
Determinant

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Motivation

- 雖然求行列式值（determinant）的方法，大多數同學，高中時就學過了。可是我們這裡還是簡單的複習一下，順便看看行列式值有哪些特性？有哪些應用？
- 本講稿只是大略的點出一些概念，詳細的推導，將留給各位同學當作習題，練習練習！

Applications of Determinant

1. Find A^{-1} analytically.
2. Solve A**x**=**b** analytically. (Cramer's rule)
3. Find the size of 'volume' built by the n row vectors of the nxn matrix A in the n-dimensional space.
4. Find the eigen vlaue of the matrix A

   (Solve A**x**=l**x**)

How to evaluate the determinant of a square matrix A

- Multiply the n pivots of an nxn matrix A

  如果做了 k 次row exchange, 就要再乘以$(-1)^k$

  電腦程式用的數值方法 (the pivot formula)

- $\det(A)=S\,(e_{ijklmn\ldots}A_{1i}A_{2j}A_{3k}A_{4l}A_{5m}A_{6n}\ldots)$

  最傳統的定義（the big formula，華而不實，但包含許多行列式的特性！）

- cofactor formula (降階法)

  人工計算的常用方法

Meaning of Determinant (Volume picture)

- The absolute value of the determinant of an nxn matrix A can be considered as the ‘volume’ built by the n row vectors, or the n column vectors [since $\det(A)=\det(A^T)$], in the n-dimensional vector space.

Meaning of Determinant (surface picture 吕凌霄的看法)

- The determinant of an nxn matrix A can also be considered as the 'direction and area of the surface' in the (n+1)-dimensional vector space, where the sign of the determinant denotes the direction of the surface and the absolute value of the determinant denotes the area of the surface.
  - The surface in the (n+1)-dimensional vector space is built by the n row vectors, or the n column vectors, of A, and with the n+1 component equal to zero.

Meaning of Determinant (surface picture 呂凌霄的看法)

- The surface picture of the determinant explains why the determinant changes sign after a row exchange.
  - The exchange of row will change the normal direction of the surface in the (n+1)-dimensional vector space.

Properties 1, 2, 3 of Determinant

1. det (**1**)=1
   - 想想看，邊長為一，且每邊互相垂直的正方形面積，或正方體體積，是多少？
2. det(A with k row exchanges)=$(-1)^k$det(A)
3. The determinant is a linear function of each row separately.
   - 想想看，要讓面積增加一倍，是不是只需要將某一邊增長一倍就夠了？

Examples of Property 3

Q: det(kA)=kdet(A) ?

A: False

$$\det\begin{bmatrix} ka & kb \\ c & d \end{bmatrix} = k\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

某一邊增長k倍，面積就增加k倍

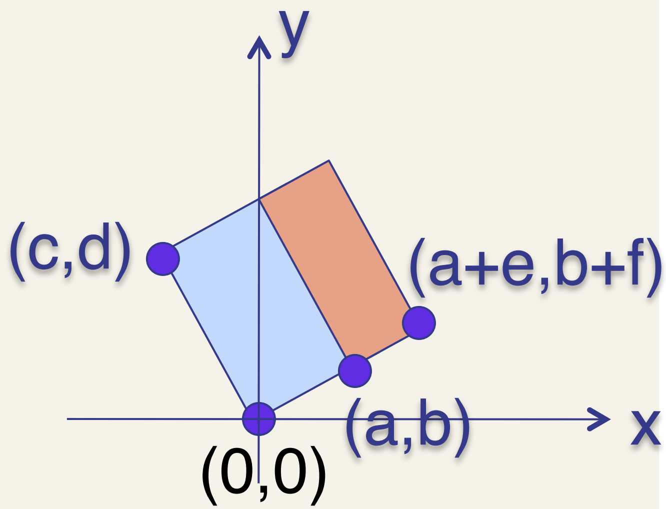
$$\text{But } k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} \text{ and } \det\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} = k^2\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

兩邊都增長k倍，面積就增加k^2倍

Examples of Property 3

Q: det(A+B)=det(A)+det(B) ?

A: False



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ c & d \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+c & d+d \end{bmatrix}$$

$$\text{but } \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \det\begin{bmatrix} e & f \\ c & d \end{bmatrix} = \det\begin{bmatrix} a+e & b+f \\ c & d \end{bmatrix} \neq \det\begin{bmatrix} a+e & b+f \\ c+c & d+d \end{bmatrix}$$

某一邊增加一段，面積就是原來的一塊，再加上另外一塊。

Properties 4~6 of Determinant

4. If two rows of A are equal, then det(A)=0
5. Subtracting a multiple of one row from another row leaves det(A) unchanged.

$$\det\begin{bmatrix} a & b \\ c-ka & d-kb \end{bmatrix} \underset{\text{property 3}}{=} \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} - k\det\begin{bmatrix} a & b \\ a & b \end{bmatrix} \underset{\text{property 4}}{=} \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} + 0$$

6. If a row of A are zeros, then det(A)=0

Properties 7~10 of Determinant

7. If A is upper or lower triangular matrix,
   then det(A)=product of the diagonal entries
8. If A is singular then det(A)=0.
   If A is invertible then ldet(A)l >0
9. det(AB)=det(A)det(B)
   →det(A)=1/det(A^{-1}) (if A is invertible)
10. det(A)=det(A^T)

Pivot Formula

Q: det(elimination matrix E)=?

Q: det(permutation matrix P)=?

Q: det($E_3E_2E_1A$)=?

Your home works

Cramer's Rule

A **x** **=b** $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ can be written as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1 & 0 \\ x_2 & 1 \end{bmatrix}=\begin{bmatrix} b_1 & A_{12} \\ b_2 & A_{22} \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} 1 & x_1 \\ 0 & x_2 \end{bmatrix}=\begin{bmatrix} A_{11} & b_1 \\ A_{21} & b_2 \end{bmatrix}$$

Making use of the property 10:

det(AB)=det(A)det(B), one can solve **x**=?

Your home work

Using determinant of A to determine A^{-1}

Let A be a 3x3 matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)}\left[(\text{cofactors})_{\text{ij}}\right]^T$$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} \det\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} & -\det\begin{bmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{bmatrix} & \det\begin{bmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \\ -\det\begin{bmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{bmatrix} & \det\begin{bmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{bmatrix} & -\det\begin{bmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{bmatrix} \\ \det\begin{bmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{bmatrix} & -\det\begin{bmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{bmatrix} & de\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{bmatrix}^T$$

Using determinant of A to determine A^{-1}

50% students forgot to take transpose!

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)}\left[(\text{cofactors})_{\text{ij}}\right]^T$$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} \det\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} & -\det\begin{bmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{bmatrix} & \det\begin{bmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{bmatrix} \\ -\det\begin{bmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{bmatrix} & \det\begin{bmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{bmatrix} & -\det\begin{bmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{bmatrix} \\ \det\begin{bmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} & -\det\begin{bmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{bmatrix} & de\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{bmatrix}$$

Using determinant of A to determine A^{-1}

- Prove that AB=**1**, where

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Your home work

$$B = \frac{1}{\det(A)} \begin{bmatrix} \det\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} & -\det\begin{bmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{bmatrix} & \det\begin{bmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{bmatrix} \\ -\det\begin{bmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{bmatrix} & \det\begin{bmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{bmatrix} & -\det\begin{bmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{bmatrix} \\ \det\begin{bmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} & -\det\begin{bmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{bmatrix} & de\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{bmatrix}$$

Selected Problems

- Section 5.1
  - Problems 1,5,8,10,11,18,27 (pp. 240-243)
- Section 5.2
  - Problems 20, 30, 36 (pp. 256-259)
- Section 5.3
  - What is the Jacobin matrix? (p. 256)
  - Problems 6, 21,27-31(pp. 270-272)