太空科學研究所

Linear Algebra
Lecture 3 (Chap. 4)
Projection and Projection Matrix

Ling-Hsiao Lyu

Institute of Space Science, National Central University

Chung-Li, Taiwan, R. O. C.

Projection to a Line

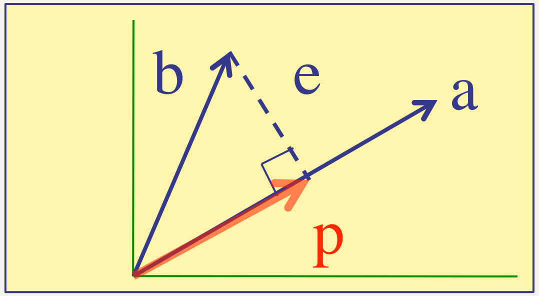
Projection Matrix P projects vector **b** to **a**.

Let $P\mathbf{b} = \xi \mathbf{a} = \mathbf{p}, \quad error: \mathbf{e} = \mathbf{b} - \mathbf{p}, \quad \mathbf{a} \perp \mathbf{e} \Rightarrow \mathbf{a}^T \mathbf{e} = 0$

$$\mathbf{a}^T \mathbf{e} = 0 = \mathbf{a}^T (\mathbf{b} - \mathbf{p}) = \mathbf{a}^T (\mathbf{b} - P\mathbf{b}) = \mathbf{a}^T (\mathbf{b} - \xi \mathbf{a})$$

$$\xi \text{ is a scalar } \Rightarrow \xi \mathbf{a}^T \mathbf{a} = \mathbf{a}^T \mathbf{b} \quad \Rightarrow \boxed{\xi = \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}}} \left(= \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right)$$

$$P\mathbf{b} = \xi \mathbf{a} = \mathbf{a} \xi = \mathbf{a} \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}} = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T \mathbf{a}} \mathbf{b} \quad \Rightarrow \boxed{P = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T \mathbf{a}}}$$



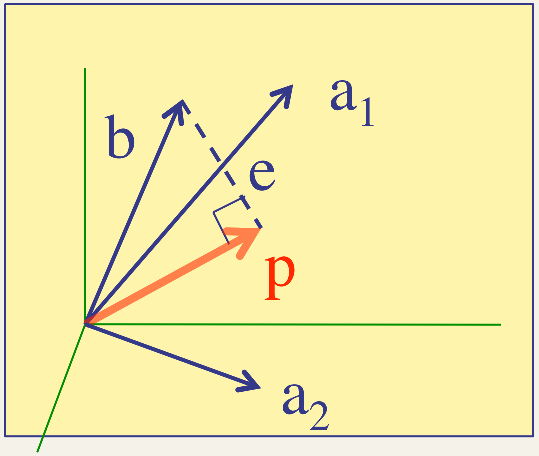
Projection to a Plane or to an N-dimensional Space

Projection Matrix P projects **b** into the column space of A. Let

$$P\mathbf{b} = \mathbf{p} \in C(A)$$
$$\mathbf{b} - \mathbf{p} = \mathbf{e} \in N(A^T)$$

$$A = \begin{bmatrix} \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 \\ \downarrow & \downarrow \end{bmatrix}, \quad P\mathbf{b} = x_1 \begin{bmatrix} \uparrow \\ \mathbf{a}_1 \\ \downarrow \end{bmatrix} + x_2 \begin{bmatrix} \uparrow \\ \mathbf{a}_2 \\ \downarrow \end{bmatrix} = A\hat{\mathbf{x}} = \mathbf{p}, \quad \mathbf{e} = \mathbf{b} - \mathbf{p}, \quad A^T\mathbf{e} = 0$$

$$A^T\mathbf{e} = 0 = A^T(\mathbf{b} - \mathbf{p}) = A^T(\mathbf{b} - P\mathbf{b}) = A^T(\mathbf{b} - A\hat{\mathbf{x}})$$
$$\Rightarrow A^T A\hat{\mathbf{x}} = A^T\mathbf{b} \quad \Rightarrow \boxed{\hat{\mathbf{x}} = (A^T A)^{-1} A^T\mathbf{b}}$$
$$P\mathbf{b} = A\hat{\mathbf{x}} = A(A^T A)^{-1} A^T\mathbf{b} \quad \Rightarrow \boxed{P = A(A^T A)^{-1} A^T}$$



Projection Matrix: $P=P^T=P^2$

- $P^T=P$

$$P=\frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} \Rightarrow P^T=\frac{(\mathbf{a}\mathbf{a}^T)^T}{\mathbf{a}^T\mathbf{a}}=\frac{(\mathbf{a}^T)^T(\mathbf{a})^T}{\mathbf{a}^T\mathbf{a}}=\frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}=P$$

$$P=A(A^TA)^{-1}A^T \Rightarrow P^T=(A(A^TA)^{-1}A^T)^T=(A^T)^T[(A^TA)^{-1}]^TA^T$$
$$=A(A^T(A^T)^T)^{-1}A^T=A(A^TA)^{-1}A^T=P$$

- $P^2=P$

$$P=\frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} \Rightarrow P^2=\frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}\frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}=\frac{\mathbf{a}(\mathbf{a}^T\mathbf{a})\mathbf{a}^T}{(\mathbf{a}^T\mathbf{a})^2}=\frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}=P$$

$$P=A(A^TA)^{-1}A^T \Rightarrow P^2=(A(A^TA)^{-1}A^T)(A(A^TA)^{-1}A^T)$$
$$=A(A^TA)^{-1}\,I\,A^T=A(A^TA)^{-1}A^T=P$$

- Show that $(M^T)^{-1}=(M^{-1})^T$ where M is an $n\times n$ matrix

Example 1

- Find the projection of vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ on vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\mathbf{p} = \frac{\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{-1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

- Find the projection matrix P that projects any given vector in R^2 to the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$P = \frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix}}{\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Example 2

- Find (a) the projection of vector $\mathbf{b}=\begin{bmatrix}1\\1\\1\end{bmatrix}$ on the column space of matrix $A=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}$

and (b) the projection matrix P that projects any vector in R^3 to the C(A).

Answer : There are two ways to determine projection vector **p**.

Method 1: Determine the coefficient vector $\hat{\mathbf{x}}$ based on $A^T\mathbf{e}=0$, then determine $\mathbf{p}$ from $\mathbf{p}=A\hat{\mathbf{x}}$.

$$A^T\mathbf{e}=0=A^T(\mathbf{b}-\mathbf{p})=A^T(\mathbf{b}-A\hat{\mathbf{x}})\Rightarrow A^T\mathbf{b}=A^TA\hat{\mathbf{x}}$$

$$\Rightarrow \hat{\mathbf{x}}=(A^TA)^{-1}A^T\mathbf{b}=(\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix})^{-1}\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=(\begin{bmatrix}2&1\\1&2\end{bmatrix})^{-1}\begin{bmatrix}2\\2\end{bmatrix}=\frac{1}{3}\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix}=\frac{1}{3}\begin{bmatrix}2\\2\end{bmatrix}\Rightarrow \mathbf{p}=A\hat{\mathbf{x}}=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}\frac{1}{3}\begin{bmatrix}2\\2\end{bmatrix}=\frac{1}{3}\begin{bmatrix}2\\4\\2\end{bmatrix}=\boxed{\begin{bmatrix}2/3\\4/3\\2/3\end{bmatrix}}$$

(a)

Method 2: Find the projection matrix P first, then determine $\mathbf{p}$ from $\mathbf{p}=P\mathbf{b}$.

$$P=A(A^TA)^{-1}A^T=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}(\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix})^{-1}\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}(\frac{1}{3}\begin{bmatrix}2&-1\\-1&2\end{bmatrix})\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}2&-1\\1&1\\-1&2\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}=\boxed{\frac{1}{3}\begin{bmatrix}2&1&-1\\1&2&1\\-1&1&2\end{bmatrix}}$$

(b)

$$\mathbf{p}=P\mathbf{b}=\frac{1}{3}\begin{bmatrix}2&1&-1\\1&2&1\\-1&1&2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}2\\4\\2\end{bmatrix}=\boxed{\begin{bmatrix}2/3\\4/3\\2/3\end{bmatrix}}$$

(a)

Always use at least two different methods to verify your results