太空科學研究所

Linear Algebra
Lecture 2 (Chap. 3)
General Solution of Ax=b
and Vector Space

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Outlines

- Solving $A\mathbf{x}=0$
- General solution of $A\mathbf{x}=\mathbf{b}$, $\mathbf{x}=\mathbf{x}_n+\mathbf{x}_p$
- Vector Space
  - Column Space
  - Row Space
  - Null Space
- Solving $A\mathbf{x}_p=\mathbf{b}$ with $\mathbf{x}_p$ perpendicular to $\mathbf{x}_n$.

Solving Linear Equations

$$A_{mxn}\mathbf{x}_{nx1}=\mathbf{b}_{mx1}$$

The general solution of $A\mathbf{x}=\mathbf{b}$ is

$$\mathbf{x}=\mathbf{x}_n+\mathbf{x}_p$$

where $A\mathbf{x}_n=0$ and $A\mathbf{x}_p=\mathbf{b}$.

Step-by-Step Solving Ax=0 (non-trivial solution)

- Let $A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$. Solve A**x**=0.
- Reduce A to a row echelon form

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step-by-Step Solving Ax=0 (non-trivial solution)

- Solving A**x**=0 is the same as solving

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

pivot variables

free variables

解法：
設定free variables所對應之x_i值為identity matrix每一列的形式。也就是如果只有一個free variable 就設它為1。如果如本例，有兩個free variables，則有兩組通解。因此free variables所對應之(x_2, x_4)的通解分別為(1,0) and (0,1). 再根據此通解，帶入求x_3, x_1.

Step-by-Step Solving Ax=0 (non-trivial solution)

- Solving A**x**=0 is the same as solving

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

pivot variables (x_1, x_3) — free variables (x_2, x_4)

解法:
$(x_2, x_4)=(1,0) \rightarrow x_3=0, x_1= -2.$
$(x_2, x_4)=(0,1) \rightarrow x_3= -2, x_1=2.$
故

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Step-by-Step Solving Ax=0 (non-trivial solution)

- Solving A**x**=0 is the same as solving

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

pivot variables

free variables

另一種看法：由 $2x_3 + 4x_4 = 0$
可以看出，一個方程式有兩個未知數。所以可以隨便猜一個x_4,然後反求x_3。一但x_3x_4的值確定了，以下的方程式，

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$

就只剩兩個變數x_1x_2，又是，一個方程式有兩個未知數。同法可以隨便猜一個x_2,然後反求x_1。

Step-by-Step Solving Ax=0 (non-trivial solution)

猜x_4與x_2的方法：可以猜它們 等於零 或 不等於零。
其中，不等於零的值，最簡單的就是1。
但是如果想猜一個異於1的值也是可以的。

最後，2x2，就有4種排列組合。

其中$(x_2,x_4)=(0,0)$之解，(x_1,x_3)也是零。
因此**x**的解為trivial solution，可以忽略不計。

另一個$(x_2,x_4)=(1,1)$，所求出的**x**解，可寫為$(x_2,x_4)=(1,0)$與$(x_2,x_4)= (0,1)$所求出的**x**解的線性組合。因此也可以忽略不計。

Solving Ax=0 (non-trivial solution)

- Reduce A to "reduced row echelon form" rref

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \rightarrow \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} = 0$$

這就是MATLAB如何找出A**x**=0之解的方法！

$$\begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} I_{2\times 2} & F_{2\times 2} \\ 0_{1\times 2} & 0_{1\times 2} \end{bmatrix} \begin{bmatrix} -F_{2\times 2} \\ I_{2\times 2} \end{bmatrix} = 0$$

Step-by-Step Solving A**x**=**b**

- General solution of A**x**=**b** is $\mathbf{x}=\mathbf{x}_n+\mathbf{x}_p$

  where $A\mathbf{x}_n=0$ and $A\mathbf{x}_p=\mathbf{b}$

$\mathbf{x}_n$解法:

設定free variables所對應之x_{fi}值為identity matrix每一列的形式。也就是如果只有一個free variable 就設它為1。如果，有兩個free variables，則有兩組通解。free variables所對應之(x_{f1},x_{f2})通解分別為(1,0) and (0,1). 再根據此通解，帶入求支點變數解x_{pj}.

$\mathbf{x}_p$解法:

設定free variables所對應之x_{fi}值為0。帶入求支點變數解x_{pj}. 組合起來的解就是$\mathbf{x}_p$

Solving Ax=0 or Ax=b

- 當未知數的個數多於方程式的個數，可有無窮多組解。這時候需要加入更多的條件（也許不是線性的方程式），才能求得唯一解！
- 例題：小明養了雞、兔、與螞蟻，共有6個頭，28隻腳，請問小明養了幾隻雞x？幾隻兔子y？與幾隻螞蟻z？

$$\text{Solve}\quad \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 28 \end{bmatrix} \quad \text{or solve} \quad \begin{bmatrix} 1 & 1 & 1 & -6 \\ 2 & 4 & 6 & -28 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

Solving $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 28 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 4 & 6 & 28 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & 4 & 16 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}_n = c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \\ z \end{bmatrix}_p = \begin{bmatrix} -2 \\ 8 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 8 \\ 0 \end{bmatrix}$$

For positive integers x, y, z, we choose $c_1 = 3 \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

其他條件！

Vector Space

If vector $\mathbf{x}=(x_1, x_2, x_3, x_4)$ then
$\mathbf{x}$ is a vector in the R^4 space.

R^4 is a vector space with dimension of 4.

Vector Space (向量空間)

A vector space is different from a set.

向量空間不同於由一些點所構成的集合。

- 在 x-y 平面上，所有滿足 3x+4y=5 的點所構成的集合是一條直線。但是只有過原點的直線3x+4y=0，才構成一個向量空間。因為…

向量空間中，任意兩個向量的線性組合，仍必須位在此向量空間中！

A+(-A)=0 故向量空間一定要包含原點。

Special Space
(特殊空間)

R^n : the vector space that consists of all column vectors with n components.

向量空間中，任意兩個向量的線性組合，仍必須位在此向量空間中！

A+(-A)=0 故向量空間一定要包含原點。

M : the matrix space

Z : the zero-dimensional space

Subspace (子空間)

Subspace is a vector space.以三度空間為例：

三度空間中過原點的一個面，是此三度空間中的一個子空間。

三度空間中過原點的一條線，也是此三度空間中的一個子空間。

向量空間中，任意兩個向量的線性組合，仍必須位在此向量空間中！

Subspace (子空間)

考慮三度空間中一個不過原點的一個面，由原點指向該面上相異的任意兩點，所構成的兩個向量之和，一定不會落在此面上，所以不滿足【**向量空間**】的定義。故不是一個子空間。

向量空間中，任意兩個向量的線性組合，仍必須位在此向量空間中！

Orthogonal Subspaces (相互垂直的子空間)

當一個子空間中的所有**向量**完全垂直於另外一個子空間中的**向量**，我們就說，這兩個子空間互相垂直（orthogonal）。

注意：所有的**向量**，都是由原點出發，指向空間中的某一點。

Orthogonal Subspaces (相互垂直的子空間)

範例一：考慮三度空間中過原點的兩個面，夾角為90度。

- 這兩個面互相垂直。
- 這兩個面，都通過原點，所以都算此三度空間中的子空間。
- 但是這兩個子空間卻不互相垂直！（Do you know why?）

Orthogonal Subspaces (相互垂直的子空間)

範例二：

三度空間中過原點的一個面，與垂直此面且過原點的一直線。兩者都算此三度空間中的子空間。而且這兩個子空間互相垂直！ 注意：

平面的dimension為2，直線的dimension為1。

2+1=3，剛好是完整向量空間的dimension值。

Orthogonal Subspaces (相互垂直的子空間)

- 完整向量空間中，相互垂直的子空間的維數和，等於完整向量空間的維數。
- 完整向量空間的維數（dimension）由向量表示式的分量個數決定之。

例如：向量 $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ 有三個分量，所以是一個三度空間中的向量。

Four Fundamental Subspaces of a Matrix A

Let A be an mxn matrix

C(A) : The Column Space of matrix A is the vector space spanned by the column vectors of matrix A. (i.e., all linear combinations of the column vectors)

Since each column vector has m components, C(A) is a subspace of $\mathbf{R}^m$

Four Fundamental Subspaces of a Matrix A

Let A be an mxn matrix

R(A) : The Row Space of matrix A is the vector space spanned by the row vectors of matrix A. Since each row vector has n components, R(A) is a subspace of $\mathbf{R}^n$

Four Fundamental Subspaces of a Matrix A

Let A be an mxn matrix

N(A) : The Nullspace of matrix A is the vector space spanned by all the vectors **x** which satisfy A**x**=0. Since the vector **x** has n components, N(A) is a subspace of $\mathbf{R}^n$.

N(A) and R(A) are two orthogonal subspaces of the space $\mathbf{R}^n$.

Four Fundamental Subspaces of a Matrix A

Let A be an mxn matrix

$N(A^T)$: The Left Nullspace of matrix A, or the nullspace of A^T, is the vector space spanned by all the vectors **y** which satisfy $A^T\mathbf{y}=0$. Since the vector **y** has m components, $N(A^T)$ is a subspace of $\mathbf{R}^m$.

$N(A^T)$ and C(A) are two orthogonal subspaces of the space $\mathbf{R}^m$.

Rank of a matrix

- A is an $m \times n$ matrix. Let r be the rank of matrix A. We have $r \le m$ and $r \le n$

- Example:

$$A\mathbf{x} = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 5 \\ 1 & 3 & 4 \\ -2 & 4 & 2 \\ 1 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Since Col.1+Col.2=Col.3, there are only two independent column vectors

- The rank of matrix A is 2.

Dimension of a space or subspace

- Example:

$$A\mathbf{x} = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 5 \\ 1 & 3 & 4 \\ -2 & 4 & 2 \\ 1 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Since Col.1+Col.2=Col.3, there are only two independent column vectors

The rank of matrix A is 2.

The dimension of column space C(A)=2

The dimension of row space R(A) = 2

Four Fundamental Subspaces (example)

Example:

$$A\mathbf{x} = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 5 \\ 1 & 3 & 4 \\ -2 & 4 & 2 \\ 1 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$A^T\mathbf{y} = \begin{bmatrix} 1 & 3 & 1 & -2 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 2 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = 0$$

Dimension of the Four Fundamental Subspaces

A is a 5x3 matrix. The rank of matrix A = 2.

The dimension of C(A) = 2

The dimension of R(A) = 2

The C(A) is a subspace of R^5, so is $N(A^T)$.

The R(A) is a subspace of R^3, so is N(A).

The dimension of N(A) is 3-2=1

The dimension of $N(A^T)$=5-2=3

Definitions (1)

- Rank
  - Number of independent components
  - Degrees of freedom
- Rank of a matrix
- Number of independent equations
- Dimension of a vector space
- Span
  - N independent vectors can span a vector space with dimension N

Definitions (2)

- Linear combinations $c\mathbf{u} + d\mathbf{v}$
- Linear transformation

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

- Linear transformation can be represented by a matrix

Definitions(3)

- A linear transformation f , that transform a vector from V to a scalar field, is also called a linear functional on V .
- The dual space: a collection of all linear functionals on V can form a vector space V*, which is called the dual space of V.

Solving $A\mathbf{x}_p=\mathbf{b}$ with $\mathbf{x}_p$ perpendicular to $\mathbf{x}_n$

- The general solution of $A\mathbf{x}=\mathbf{b}$ is $\mathbf{x}=\mathbf{x}_n+\mathbf{x}_p$,where $A\mathbf{x}_n=0$, $A\mathbf{x}_p=\mathbf{b}$, and $\mathbf{x}_p$ is independent of $\mathbf{x}_n$.
- $\mathbf{x}_p$ 可以完全位在 row space of A，也可以超出 row space of A. 若 $\mathbf{x}_p$完全位在 R(A)中，則 $\mathbf{x}_p$ is a linear combination of all the row vectors of the matrix A and $\mathbf{x}_p$ is perpendicular to $\mathbf{x}_n$.
- The Gaussian elimination yields $L^{-1}A=L^{-1}LU=U$ and $L^{-1}\mathbf{b}=\mathbf{c}$ then we have $U\mathbf{x}=\mathbf{c}$, $U\mathbf{x}_n=0$, and $\mathbf{x}_p$ is also a linear combination of all the row vectors of the matrix U.

Ax_p=b

x_p is a linear combination of all the row vectors of the matrix A

- If $A = \begin{bmatrix} \leftarrow & a_1 & \rightarrow \\ \leftarrow & a_2 & \rightarrow \\ \leftarrow & a_3 & \rightarrow \end{bmatrix}$

(In this example, if the rank of the matrix A is 3 then AA^T will be an invertible matrix)

- then

$$\mathbf{x}_p = y_1 \begin{bmatrix} \uparrow \\ a_1 \\ \downarrow \end{bmatrix} + y_2 \begin{bmatrix} \uparrow \\ a_2 \\ \downarrow \end{bmatrix} + y_3 \begin{bmatrix} \uparrow \\ a_3 \\ \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ a_1 & a_2 & a_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

- Namely, if AA^T is an invertible matrix, then

$$A\mathbf{x}_p = \mathbf{b} \Rightarrow A(A^T\mathbf{y}) = \mathbf{b} \Rightarrow \mathbf{y} = (AA^T)^{-1}\mathbf{b}$$
$$\Rightarrow \mathbf{x}_p = A^T\mathbf{y} = A^T(AA^T)^{-1}\mathbf{b}$$

$U\mathbf{x}_p = \mathbf{c}$

$\mathbf{x}_p$ is a linear combination of all the row vectors of the matrix U

Likewise, if UU^T is an invertible matrix, then

$$U\mathbf{x}_p = \mathbf{c} \Rightarrow \mathbf{x}_p = U^T(UU^T)^{-1}\mathbf{c}$$

Otherwise $$U\mathbf{x}_p = \mathbf{c} \Rightarrow U(U^T\mathbf{z}) = \mathbf{c} \Rightarrow \text{Solving } \mathbf{z} \Rightarrow \mathbf{x}_p = U^T\mathbf{z}$$

- 注意：相較之下，前面第十頁中 A$\mathbf{x}_p$=**b**的解法只保證 $\mathbf{x}_p$ is independent of $\mathbf{x}_n$ 但無法保證 $\mathbf{x}_p$ is perpendicular to $\mathbf{x}_n$ (i.e., $\mathbf{x}_p$ may not be a linear combination of all the row vectors of the matrix U.

Summary: Figures

on pages 176,185, 188, 396, 413 and the back cover of the textbook

- Four fundamental subspaces of an mxn matrix A