Let

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \end{bmatrix} \tag{1}$$

Given

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ and } \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

Find the general solution of

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Answer:

The transport of equation (1) yields

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} d & e & f \end{bmatrix} \tag{2}$$

Equation (2) can be written as

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} A_{11} \\ A_{12} \\ A_{13} \end{bmatrix} = [d] \tag{3}$$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} A_{21} \\ A_{22} \\ A_{23} \end{bmatrix} = [e] \tag{4}$$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} A_{31} \\ A_{32} \\ A_{33} \end{bmatrix} = [f] \tag{5}$$

Note that we have 3 equations but 9 unknowns. There must be 6 free parameters in the general solution.

Let us first consider the general solution of equation (3).

Let

$$\mathbf{x} = \begin{bmatrix} A_{11} \\ A_{12} \\ A_{13} \end{bmatrix} = \mathbf{x}_n + \mathbf{x}_p$$

where

$\begin{bmatrix} a & b & c \end{bmatrix}\mathbf{x}_n = 0_{1\times1}$ and $\begin{bmatrix} a & b & c \end{bmatrix}\mathbf{x}_p = [d]$

Since $\begin{bmatrix} a & b & c \end{bmatrix} \in R^3$, the dimension of the null space of $\begin{bmatrix} a & b & c \end{bmatrix}$ must be $3-1=2$. Thus, we consider the following null solutions

$$\begin{bmatrix} 1 & \frac{b}{a} & \frac{c}{a} \end{bmatrix} \begin{bmatrix} -\frac{b}{a} & -\frac{c}{a} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = 0_{1\times2} \text{. That is } \mathbf{x}_n = c_1 \begin{bmatrix} -\frac{b}{a} \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -\frac{c}{a} \\ 0 \\ 1 \end{bmatrix}$$

and particular solutions

$$\begin{bmatrix} 1 & \frac{b}{a} & \frac{c}{a} \end{bmatrix} \begin{bmatrix} \frac{d}{a} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{d}{a} \end{bmatrix}\text{. That is } \mathbf{x}_p = \begin{bmatrix} \frac{d}{a} \\ 0 \\ 0 \end{bmatrix}$$

It yields one possible solutions of equation (3) is

$$\begin{bmatrix} A_{11} \\ A_{12} \\ A_{13} \end{bmatrix} = c_1 \begin{bmatrix} -\frac{b}{a} \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -\frac{c}{a} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{d}{a} \\ 0 \\ 0 \end{bmatrix} \tag{6a}$$

It can be shown that there are two other possible solution forms for equation (3). That is

$$\begin{bmatrix} A_{11} \\ A_{12} \\ A_{13} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -\frac{a}{b} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -\frac{c}{b} \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{d}{b} \\ 0 \end{bmatrix} \tag{6b}$$

or

$$\begin{bmatrix} A_{11} \\ A_{12} \\ A_{13} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ -\frac{a}{c} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -\frac{b}{c} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{d}{c} \end{bmatrix} \tag{6c}$$

Likewise, we can find three types of general solutions for equation (4):

$$\begin{bmatrix} A_{21} \\ A_{22} \\ A_{23} \end{bmatrix} = c_3 \begin{bmatrix} -\frac{b}{a} \\ 1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} -\frac{c}{a} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{e}{a} \\ 0 \\ 0 \end{bmatrix} \tag{7a}$$

or

$$\begin{bmatrix} A_{21} \\ A_{22} \\ A_{23} \end{bmatrix} = c_3 \begin{bmatrix} 1 \\ -\frac{a}{b} \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ -\frac{c}{b} \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{e}{b} \\ 0 \end{bmatrix} \tag{7b}$$

or

$$\begin{bmatrix} A_{21} \\ A_{22} \\ A_{23} \end{bmatrix} = c_3 \begin{bmatrix} 1 \\ 0 \\ -\frac{a}{c} \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 1 \\ -\frac{b}{c} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{e}{c} \end{bmatrix} \tag{7c}$$

We can also find three types of general solutions for equation (5):

$$\begin{bmatrix} A_{31} \\ A_{32} \\ A_{33} \end{bmatrix} = c_5 \begin{bmatrix} -\frac{b}{a} \\ 1 \\ 0 \end{bmatrix} + c_6 \begin{bmatrix} -\frac{c}{a} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{f}{a} \\ 0 \\ 0 \end{bmatrix} \tag{8a}$$

or

$$\begin{bmatrix} A_{31} \\ A_{32} \\ A_{33} \end{bmatrix} = c_5 \begin{bmatrix} 1 \\ -\frac{a}{b} \\ 0 \end{bmatrix} + c_6 \begin{bmatrix} 0 \\ -\frac{c}{b} \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{f}{b} \\ 0 \end{bmatrix} \tag{8b}$$

or

$$\begin{bmatrix} A_{31} \\ A_{32} \\ A_{33} \end{bmatrix} = c_5 \begin{bmatrix} 1 \\ 0 \\ -\frac{a}{c} \end{bmatrix} + c_6 \begin{bmatrix} 0 \\ 1 \\ -\frac{b}{c} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{f}{c} \end{bmatrix} \tag{8c}$$

The final solution can be any combination of (6a) (7a) (8a) or (6b)(7a)(8a), or ..., etc. Namely, there are at least 9 types of solutions. For example, the solution from the combination of (6a), (7a), and (8a), can be written as

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} (-c_1\frac{b}{a} - c_2\frac{c}{a} + \frac{d}{a}) & c_1 & c_2 \\ (-c_3\frac{b}{a} - c_4\frac{c}{a} + \frac{e}{a}) & c_3 & c_4 \\ (-c_5\frac{b}{a} - c_6\frac{c}{a} + \frac{f}{a}) & c_5 & c_6 \end{bmatrix} \quad (9)$$

Check the results by substituting equation (9) into (1), it yields

$$\begin{bmatrix} (-c_1\frac{b}{a} - c_2\frac{c}{a} + \frac{d}{a}) & c_1 & c_2 \\ (-c_3\frac{b}{a} - c_4\frac{c}{a} + \frac{e}{a}) & c_3 & c_4 \\ (-c_5\frac{b}{a} - c_6\frac{c}{a} + \frac{f}{a}) & c_5 & c_6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

So, equation (9) is one of the valid solutions of equation (1) and contains 6 free parameters.

In order to introduce the concept of **projection**, let us reexamine the solution of equation (3).

$$\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} A_{11} \\ A_{12} \\ A_{13} \end{bmatrix} = [d] \tag{3}$$

$$\begin{bmatrix} A_{11} \\ A_{12} \\ A_{13} \end{bmatrix} = c_1\begin{bmatrix} -\frac{b}{a} \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -\frac{c}{a} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{d}{a} \\ 0 \\ 0 \end{bmatrix} \tag{6a}$$

$$= c_1\mathbf{x}_{n1} \qquad + c_2\mathbf{x}_{n2} \qquad + \mathbf{x}_p$$

We can rewrite the equation (6a) into the following form

$$\begin{bmatrix} A_{11} \\ A_{12} \\ A_{13} \end{bmatrix} = d_1\mathbf{y}_{n1} + d_2\mathbf{y}_{n2} + \mathbf{y}_p \tag{10}$$

where $\mathbf{y}_{n1}$, $\mathbf{y}_{n2}$, $\mathbf{y}_p$ are perpendicular to each other, and both $\mathbf{y}_{n1}$ and $\mathbf{y}_{n2}$ are unit vectors. We can determine $\mathbf{y}_{n1}$, $\mathbf{y}_{n2}$, $\mathbf{y}_p$ by the following procedure.

Step 1: Find normalized $\mathbf{y}_{n1}$ from $\mathbf{x}_{n1}$

$$\mathbf{y}_{n1} = \frac{\mathbf{x}_{n1}}{\|\mathbf{x}_{n1}\|} = \frac{1}{\sqrt{1+\frac{b^2}{a^2}}}\begin{bmatrix} -\frac{b}{a} \\ 1 \\ 0 \end{bmatrix} \tag{11}$$

Step 2: Find $\mathbf{z}_{n2}$ from $\mathbf{x}_{n2}$ by removing the component of $\mathbf{x}_{n2}$ that is parallel to the $\mathbf{x}_{n1}$ or $\mathbf{y}_{n1}$. Then find the normalized $\mathbf{y}_{n2}$ from $\mathbf{z}_{n2}$

$$\mathbf{z}_{n2} = \boxed{\mathbf{x}_{n2} - \frac{\mathbf{x}_{n2}\cdot\mathbf{x}_{n1}\mathbf{x}_{n1}}{\|\mathbf{x}_{n1}\|^2}} = \mathbf{x}_{n2} - \frac{\mathbf{x}_{n1}\mathbf{x}_{n1}^T\mathbf{x}_{n2}}{\mathbf{x}_{n1}^T\mathbf{x}_{n1}} = (I - \frac{\mathbf{x}_{n1}\mathbf{x}_{n1}^T}{\mathbf{x}_{n1}^T\mathbf{x}_{n1}})\mathbf{x}_{n2} \tag{12a}$$

or

$$\mathbf{z}_{n2} = \boxed{\mathbf{x}_{n2} - \frac{\mathbf{x}_{n2}\cdot\mathbf{y}_{n1}\mathbf{y}_{n1}}{\|\mathbf{y}_{n1}\|^2}} = \mathbf{x}_{n2} - \frac{\mathbf{y}_{n1}\mathbf{y}_{n1}^T\mathbf{x}_{n2}}{\mathbf{y}_{n1}^T\mathbf{y}_{n1}} = (I - \mathbf{y}_{n1}\mathbf{y}_{n1}^T)\mathbf{x}_{n2} \tag{12b}$$

Physics (inner product) Mathematics (matrix multiplying)

That is

$$\mathbf{z}_{n2} = (I - \frac{\mathbf{x}_{n1}\mathbf{x}_{n1}^T}{\mathbf{x}_{n1}^T\mathbf{x}_{n1}})\mathbf{x}_{n2} = (I - \mathbf{y}_{n1}\mathbf{y}_{n1}^T)\mathbf{x}_{n2}$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{1+\frac{b^2}{a^2}} \begin{bmatrix} -\frac{b}{a} \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -\frac{b}{a} & 1 & 0 \end{bmatrix} \right) \begin{bmatrix} -\frac{c}{a} \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{\frac{b^2}{a^2}}{1+\frac{b^2}{a^2}} & \frac{\frac{b}{a}}{1+\frac{b^2}{a^2}} & 0 \\ \frac{\frac{b}{a}}{1+\frac{b^2}{a^2}} & 1 - \frac{1}{1+\frac{b^2}{a^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{c}{a} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{c}{a}(\frac{1}{1+\frac{b^2}{a^2}}) \\ -\frac{c}{a}\frac{\frac{b}{a}}{1+\frac{b^2}{a^2}} \\ 1 \end{bmatrix} = \frac{1}{a^2+b^2} \begin{bmatrix} -ca \\ -cb \\ a^2+b^2 \end{bmatrix}$$

and

$$\mathbf{y}_{n2} = \frac{\mathbf{z}_{n2}}{\|\mathbf{z}_{n2}\|} = \frac{1}{\sqrt{(a^2+b^2)(a^2+b^2+c^2)}} \begin{bmatrix} -ca \\ -cb \\ a^2+b^2 \end{bmatrix} \tag{13}$$

Step 3: Find $\mathbf{y}_p$ which is perpendicular to both $\mathbf{y}_{n1}$ and $\mathbf{y}_{n2}$. There are two ways to find $\mathbf{y}_p$.

Method 1 in Step 3: Find $\mathbf{y}_p$ based on the discussion given on **pages 33-34** in the power point file of **Lecture 2**. According to the discussion, we can find $\mathbf{y}_p$ as a linear combination of the $\begin{bmatrix} a & b & c \end{bmatrix}^T$. That is

$$\mathbf{y}_p = \alpha_0 \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \tag{14}$$

Since

$$\begin{bmatrix} a & b & c \end{bmatrix} \mathbf{y}_p = \begin{bmatrix} a & b & c \end{bmatrix} \alpha_0 \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_0 (a^2+b^2+c^2) = d \tag{15}$$

Equation (15) yields

$$\alpha_0 = \frac{d}{a^2+b^2+c^2}. \tag{16}$$

Substituting equation (16) into equation (14), it yields,

$$\mathbf{y}_p = \frac{d}{a^2+b^2+c^2}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{17}$$

Method 2 in Step 3: Find $\mathbf{y}_p$ from $\mathbf{x}_p$ by removing the projection component of $\mathbf{x}_p$ on the subspace spanned by $\mathbf{y}_{n1}$ and $\mathbf{y}_{n2}$, which is also the subspace spanned by $\mathbf{x}_{n1}$ and $\mathbf{x}_{n2}$. According to the power point file of **Lecture 3**, the projection component of $\mathbf{x}_p$ on the subspace spanned by $\mathbf{x}_{n1}$ and $\mathbf{x}_{n2}$ must be a linear combination of $\mathbf{x}_{n1}$ and $\mathbf{x}_{n2}$. Thus, $\mathbf{x}_p$ can be decomposed into the following three components

$$\mathbf{x}_p = \mathbf{y}_p + \alpha_1\mathbf{x}_{n1} + \alpha_2\mathbf{x}_{n2} \tag{18}$$

or

$$\mathbf{x}_p = \mathbf{y}_p + \beta_1\mathbf{y}_{n1} + \beta_2\mathbf{y}_{n2} \tag{19}$$

Let us first consider the equation (18)

$$\begin{bmatrix} \uparrow \\ \mathbf{x}_p \\ \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow \\ \mathbf{y}_p \\ \downarrow \end{bmatrix} + \alpha_1\begin{bmatrix} \uparrow \\ \mathbf{x}_{n1} \\ \downarrow \end{bmatrix} + \alpha_2\begin{bmatrix} \uparrow \\ \mathbf{x}_{n2} \\ \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow \\ \mathbf{y}_p \\ \downarrow \end{bmatrix} + \begin{bmatrix} \uparrow & \uparrow \\ \mathbf{x}_{n1} & \mathbf{x}_{n2} \\ \downarrow & \downarrow \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{20}$$

Multiplying $\begin{bmatrix} \leftarrow & \mathbf{x}_{n1} & \rightarrow \\ \leftarrow & \mathbf{x}_{n2} & \rightarrow \end{bmatrix}$ on the left, it yields

$$\begin{bmatrix} \leftarrow & \mathbf{x}_{n1} & \rightarrow \\ \leftarrow & \mathbf{x}_{n2} & \rightarrow \end{bmatrix}\begin{bmatrix} \uparrow \\ \mathbf{x}_p \\ \downarrow \end{bmatrix} = \begin{bmatrix} \leftarrow & \mathbf{x}_{n1} & \rightarrow \\ \leftarrow & \mathbf{x}_{n2} & \rightarrow \end{bmatrix}\begin{bmatrix} \uparrow \\ \mathbf{y}_p \\ \downarrow \end{bmatrix} + \begin{bmatrix} \leftarrow & \mathbf{x}_{n1} & \rightarrow \\ \leftarrow & \mathbf{x}_{n2} & \rightarrow \end{bmatrix}\begin{bmatrix} \uparrow & \uparrow \\ \mathbf{x}_{n1} & \mathbf{x}_{n2} \\ \downarrow & \downarrow \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$= \begin{bmatrix} \leftarrow & \mathbf{x}_{n1} & \rightarrow \\ \leftarrow & \mathbf{x}_{n2} & \rightarrow \end{bmatrix}\begin{bmatrix} \uparrow & \uparrow \\ \mathbf{x}_{n1} & \mathbf{x}_{n2} \\ \downarrow & \downarrow \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{21}$$

where $\begin{bmatrix} \leftarrow & \mathbf{x}_{n1} & \rightarrow \\ \leftarrow & \mathbf{x}_{n2} & \rightarrow \end{bmatrix}\begin{bmatrix} \uparrow \\ \mathbf{y}_p \\ \downarrow \end{bmatrix} = 0$, because we wish to find $\mathbf{y}_p$ which is perpendicular to the subspace spanned by $\mathbf{x}_{n1}$ and $\mathbf{x}_{n2}$.

Equation (21) yields

$$\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \left(\begin{bmatrix}\leftarrow & \mathbf{x}_{n1} & \rightarrow\\ \leftarrow & \mathbf{x}_{n2} & \rightarrow\end{bmatrix}\begin{bmatrix}\uparrow & \uparrow\\ \mathbf{x}_{n1} & \mathbf{x}_{n2}\\ \downarrow & \downarrow\end{bmatrix}\right)^{-1}\begin{bmatrix}\leftarrow & \mathbf{x}_{n1} & \rightarrow\\ \leftarrow & \mathbf{x}_{n2} & \rightarrow\end{bmatrix}\begin{bmatrix}\uparrow\\ \mathbf{x}_p\\ \downarrow\end{bmatrix} \qquad (22)$$

Substituting equation (22) into equation (20), it yields

$$\begin{bmatrix}\uparrow\\ \mathbf{y}_p\\ \downarrow\end{bmatrix} = \begin{bmatrix}\uparrow\\ \mathbf{x}_p\\ \downarrow\end{bmatrix} - \begin{bmatrix}\uparrow & \uparrow\\ \mathbf{x}_{n1} & \mathbf{x}_{n2}\\ \downarrow & \downarrow\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}$$

$$= \begin{bmatrix}\uparrow\\ \mathbf{x}_p\\ \downarrow\end{bmatrix} - \begin{bmatrix}\uparrow & \uparrow\\ \mathbf{x}_{n1} & \mathbf{x}_{n2}\\ \downarrow & \downarrow\end{bmatrix}\left(\begin{bmatrix}\leftarrow & \mathbf{x}_{n1} & \rightarrow\\ \leftarrow & \mathbf{x}_{n2} & \rightarrow\end{bmatrix}\begin{bmatrix}\uparrow & \uparrow\\ \mathbf{x}_{n1} & \mathbf{x}_{n2}\\ \downarrow & \downarrow\end{bmatrix}\right)^{-1}\begin{bmatrix}\leftarrow & \mathbf{x}_{n1} & \rightarrow\\ \leftarrow & \mathbf{x}_{n2} & \rightarrow\end{bmatrix}\begin{bmatrix}\uparrow\\ \mathbf{x}_p\\ \downarrow\end{bmatrix}$$

$$= [I - X_n (X_n^T X_n)^{-1} X_n^T]\mathbf{x}_p \qquad (23)$$

where

$$X_n = \begin{bmatrix}\uparrow & \uparrow\\ \mathbf{x}_{n1} & \mathbf{x}_{n2}\\ \downarrow & \downarrow\end{bmatrix}$$

and $X_n (X_n^T X_n)^{-1} X_n$ is a projection matrix that map a vector to a subspace spanned by the column vectors of matrix X_n. I will leave the evaluation of $\mathbf{y}_p$ in equation (23) to the students as a **Homework**. Let us find the solution of $\mathbf{y}_p$ from the equation (19).

$$\begin{bmatrix}\uparrow\\ \mathbf{x}_p\\ \downarrow\end{bmatrix} = \begin{bmatrix}\uparrow\\ \mathbf{y}_p\\ \downarrow\end{bmatrix} + \beta_1\begin{bmatrix}\uparrow\\ \mathbf{y}_{n1}\\ \downarrow\end{bmatrix} + \beta_2\begin{bmatrix}\uparrow\\ \mathbf{y}_{n2}\\ \downarrow\end{bmatrix} = \begin{bmatrix}\uparrow\\ \mathbf{y}_p\\ \downarrow\end{bmatrix} + \begin{bmatrix}\uparrow & \uparrow\\ \mathbf{y}_{n1} & \mathbf{y}_{n2}\\ \downarrow & \downarrow\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} \qquad (24)$$

Multiplying $\begin{bmatrix}\leftarrow & \mathbf{y}_{n1} & \rightarrow\\ \leftarrow & \mathbf{y}_{n2} & \rightarrow\end{bmatrix}$ on the left, it yields

$$\begin{bmatrix}\leftarrow & \mathbf{y}_{n1} & \rightarrow\\ \leftarrow & \mathbf{y}_{n2} & \rightarrow\end{bmatrix}\begin{bmatrix}\uparrow\\ \mathbf{x}_p\\ \downarrow\end{bmatrix} = \begin{bmatrix}\leftarrow & \mathbf{y}_{n1} & \rightarrow\\ \leftarrow & \mathbf{y}_{n2} & \rightarrow\end{bmatrix}\begin{bmatrix}\uparrow\\ \mathbf{y}_p\\ \downarrow\end{bmatrix} + \begin{bmatrix}\leftarrow & \mathbf{y}_{n1} & \rightarrow\\ \leftarrow & \mathbf{y}_{n2} & \rightarrow\end{bmatrix}\begin{bmatrix}\uparrow & \uparrow\\ \mathbf{y}_{n1} & \mathbf{y}_{n2}\\ \downarrow & \downarrow\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}$$

$$= \begin{bmatrix}\leftarrow & \mathbf{y}_{n1} & \rightarrow\\ \leftarrow & \mathbf{y}_{n2} & \rightarrow\end{bmatrix}\begin{bmatrix}\uparrow & \uparrow\\ \mathbf{y}_{n1} & \mathbf{y}_{n2}\\ \downarrow & \downarrow\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} \qquad (25)$$

Equation (25) yields

$$\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \leftarrow & \mathbf{y}_{n1} & \rightarrow \\ \leftarrow & \mathbf{y}_{n2} & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \mathbf{x}_p \\ \downarrow \end{bmatrix} \tag{26}$$

Substituting equation (26) into equation (24), it yields

$$\begin{bmatrix} \uparrow \\ \mathbf{y}_p \\ \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow \\ \mathbf{x}_p \\ \downarrow \end{bmatrix} - \begin{bmatrix} \uparrow & \uparrow \\ \mathbf{y}_{n1} & \mathbf{y}_{n2} \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & \mathbf{y}_{n1} & \rightarrow \\ \leftarrow & \mathbf{y}_{n2} & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \mathbf{x}_p \\ \downarrow \end{bmatrix} = (I - Y_n Y_n^T)\mathbf{x}_p \tag{27}$$

where $Y_n = \begin{bmatrix} \uparrow & \uparrow \\ \mathbf{y}_{n1} & \mathbf{y}_{n2} \\ \downarrow & \downarrow \end{bmatrix}$

Can you see the differences between equations (23) and (27)? This is the reason why we want to find a normalized and perpendicular set of vectors in the subspace spanned by $\mathbf{x}_{n1}$ and $\mathbf{x}_{n2}$. Let us recall the following two equations obtained before,

$$\mathbf{y}_{n1} = \frac{\mathbf{x}_{n1}}{\|\mathbf{x}_{n1}\|} = \frac{1}{\sqrt{1+\frac{b^2}{a^2}}} \begin{bmatrix} -\frac{b}{a} \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{a^2+b^2}} \begin{bmatrix} -b \\ a \\ 0 \end{bmatrix} \tag{11}$$

$$\mathbf{y}_{n2} = \frac{\mathbf{z}_{n2}}{\|\mathbf{z}_{n2}\|} = \frac{1}{\sqrt{(a^2+b^2)(a^2+b^2+c^2)}} \begin{bmatrix} -ca \\ -cb \\ a^2+b^2 \end{bmatrix} \tag{13}$$

Substituting equations (11) and (13) into equation (27), it yields

$$\begin{bmatrix} \uparrow \\ \mathbf{y}_p \\ \downarrow \end{bmatrix} = (I - Y_n Y_n^T)\mathbf{x}_p = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{-b}{\sqrt{a^2+b^2}} & \frac{-ca}{\sqrt{(a^2+b^2)(a^2+b^2+c^2)}} \\ \frac{a}{\sqrt{a^2+b^2}} & \frac{-cb}{\sqrt{(a^2+b^2)(a^2+b^2+c^2)}} \\ 0 & \frac{a^2+b^2}{\sqrt{(a^2+b^2)(a^2+b^2+c^2)}} \end{bmatrix} \right.$$

$$\left. \begin{bmatrix} \frac{-b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} & 0 \\ \frac{-ca}{\sqrt{(a^2+b^2)(a^2+b^2+c^2)}} & \frac{-cb}{\sqrt{(a^2+b^2)(a^2+b^2+c^2)}} & \frac{a^2+b^2}{\sqrt{(a^2+b^2)(a^2+b^2+c^2)}} \end{bmatrix} \right) \begin{bmatrix} d/a \\ 0 \\ 0 \end{bmatrix} \tag{28}$$

We can ignore the calculations in the last two columns in the 3x3 matrix in equation (28). Namely,

$$\begin{bmatrix} \uparrow \\ \mathbf{y}_p \\ \downarrow \end{bmatrix} = \begin{bmatrix} 1-\frac{b^2}{a^2+b^2}-\frac{c^2a^2}{(a^2+b^2)(a^2+b^2+c^2)} & * & * \\ \frac{ab}{a^2+b^2}-\frac{c^2ab}{(a^2+b^2)(a^2+b^2+c^2)} & * & * \\ \frac{ca(a^2+b^2)}{(a^2+b^2)(a^2+b^2+c^2)} & * & * \end{bmatrix} \begin{bmatrix} d/a \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a^2}{(a^2+b^2+c^2)} & * & * \\ \frac{ab}{(a^2+b^2+c^2)} & * & * \\ \frac{ca}{(a^2+b^2+c^2)} & * & * \end{bmatrix} \begin{bmatrix} d/a \\ 0 \\ 0 \end{bmatrix} = \frac{d}{(a^2+b^2+c^2)} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{29}$$

The result obtained in equation (29) is the same as the one obtained in equation (17). Namely, the result obtained from Method 2 in Step 3 is the same as the result obtained from Method 1 in Step 3. Similar result can be obtained from equation (23). (**It is your Homework!**)