Linear Algebra
Lecture 1 (Chap. 2)
Introduction to Linear Algebra and Matrix Operations

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Contents

Solving *n* Linear Equations With *n* Unknowns

- 簡單的問題：
  - 雞兔同籠問題（n=2）
  - 三個平面的交點（n=3）
- 複雜的問題：
  - Computer Tomography（ CT 電腦斷層掃描）（n: the number of pixels of the 3-D image）
  - Least Square Fit （最小平方差求回歸曲線）
  - Numerical Methods (particularly, the finite difference method 差分法)

殺雞用牛刀？

- 用簡單的問題，來說明牛刀的用法。所謂牛刀小試也！
- 了解了這些牛刀怎麼用，將來才會活用這些牛刀，用來殺複雜的問題！
- 學會如何把抽象的問題，類比為簡單的幾何問題，以幫助了解問題與解題！

Basic Concepts & Outlines

- Linear Combination: $c\mathbf{u} + d\mathbf{v}$
- Linear transformation

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

- Gauss elimination
- Gauss-Jordan elimination
- LU algorithm
- Elementary operators:
  - Elimination
  - Permutation
  - ...

Solving Linear Equations Example 1

- Matrix form: $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$
- Row picture:
  - intersection of two lines

$$2x - y = 0$$
$$-x + 2y = 3$$

- Column picture:
  - A linear combination of two vectors

$$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Solving Linear Equations
Example 2

- Matrix form: $$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$
- Row picture:
  - intersection of 3 planes

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y - z &= -1 \\ -3y + 4z &= 4 \end{aligned}$$

- Column picture:
  - A linear combination of three vectors

$$x\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

Gaussian Elimination for the Example 1

$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$$A\mathbf{x} = \mathbf{b} \Rightarrow \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$

$$\begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$L^{-1}A\mathbf{x} = L^{-1}\mathbf{b}$

$$\begin{bmatrix} 2 & -1 \\ 0 & 3/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$U\mathbf{x} = \mathbf{c}$

$\Rightarrow x_2 = 2 \Rightarrow x_1 = 1$

Since $L^{-1}A = U$

$\Rightarrow A = LU$

2/20上課時，誤指 $L^{-1}\mathbf{b}$ 為 $U\mathbf{b}$ 。感謝黃冠瀚課後的指正！

Gaussian Elimination for the Example 2

Your Home Work:

$$A\mathbf{x} = \mathbf{b} \Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

- Show that A=LU, determine L and U
- Determine **x** from U**x**=**c**, where **c**=L^{-1}**b**

The A=LU algorithm

- Gaussian elimination:

$$A\mathbf{x}=\mathbf{b}\Rightarrow(\varepsilon_k\cdots\varepsilon_2\varepsilon_1)A\mathbf{x}=(\varepsilon_k\cdots\varepsilon_2\varepsilon_1)\mathbf{b}$$

$$\Rightarrow \qquad U\mathbf{x}=(\varepsilon_k\cdots\varepsilon_2\varepsilon_1)\mathbf{b}=L^{-1}\mathbf{b}$$

Show that $A=LU$

$$(\varepsilon_k\cdots\varepsilon_2\varepsilon_1)A=U \quad\Rightarrow\quad (\varepsilon_k\cdots\varepsilon_2\varepsilon_1)^{-1}(\varepsilon_k\cdots\varepsilon_2\varepsilon_1)A=(\varepsilon_k\cdots\varepsilon_2\varepsilon_1)^{-1}U$$

Since $L^{-1}=(\varepsilon_k\cdots\varepsilon_2\varepsilon_1) \quad\Rightarrow\quad L=(\varepsilon_k\cdots\varepsilon_2\varepsilon_1)^{-1} \quad\Rightarrow\quad A=LU$

如何求 Elementary operator matrix 的反矩陣？

$$D^{-1}=\begin{bmatrix}1&0&0\\0&a&0\\0&0&1\end{bmatrix}^{-1}=\begin{bmatrix}1&0&0\\0&1/a&0\\0&0&1\end{bmatrix} \qquad E^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\b&0&1\end{bmatrix}^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\-b&0&1\end{bmatrix}$$

將二式乘以a倍 要還原 就乘以1/a倍

將三式加b倍的一式 要還原就減去b倍的一式

What is Algorithm?

- algorithm
  - a process or set of rules or steps to be followed in calculations or other problem-solving operations, esp. by a computer

Permutation

- Row exchange

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y - z &= -1 \\ -3y + 4z &= 4 \end{aligned} \quad \Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$$\begin{aligned} -x + 2y - z &= -1 \\ 2x - y &= 0 \\ -3y + 4z &= 4 \end{aligned} \quad \Rightarrow \begin{bmatrix} -1 & 2 & -1 \\ 2 & -1 & 0 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

Permutation : Row exchange

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} \qquad A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} \quad (PA)\mathbf{x} = P\mathbf{b}$$

$$\begin{bmatrix} -1 & 2 & -1 \\ 2 & -1 & 0 \\ 0 & -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

Permutation matrix 的反矩陣：若將一式與二式交換 要還原 就將兩者再交換一次

2/20上課時，第一式右式錯誤。感謝助教林筵捷的指正！

Permutation

- Column exchange

$$x\begin{bmatrix}2\\-1\\0\end{bmatrix}+y\begin{bmatrix}-1\\2\\-3\end{bmatrix}+z\begin{bmatrix}0\\-1\\4\end{bmatrix}=\begin{bmatrix}0\\-1\\4\end{bmatrix}\Rightarrow\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-3&4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\-1\\4\end{bmatrix}$$

$$y\begin{bmatrix}-1\\2\\-3\end{bmatrix}+x\begin{bmatrix}2\\-1\\0\end{bmatrix}+z\begin{bmatrix}0\\-1\\4\end{bmatrix}=\begin{bmatrix}0\\-1\\4\end{bmatrix}\Rightarrow\begin{bmatrix}-1&2&0\\2&-1&-1\\-3&0&4\end{bmatrix}\begin{bmatrix}y\\x\\z\end{bmatrix}=\begin{bmatrix}0\\-1\\4\end{bmatrix}$$

3/3上課時，-3誤植為3。感謝同學的指正！

Permutation: column exchange

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} \qquad A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

Permutation matrix 的反矩陣：
將一式與二式交換 要還原 將兩者再交換一次

$$(AP)(P^{-1}\mathbf{x}) = \mathbf{b}$$

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -1 & -1 \\ -3 & 0 & 4 \end{bmatrix}\begin{bmatrix} y \\ x \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

3/3上課時，-3誤植為3，紅色交換矩陣最下一row中1誤植為0。感謝同學們的指正！

The Algorithm for Solving Linear Equations

- Gaussian Elimination:
  - reduce to an echelon form
- Gauss-Jordon Elimination: (augmented matrix)
- The elementary matrices:
  - The elimination matrices
  - The permutation matrices
- The A=LU or A=LDU algorithm:

What is an echelon form ?

- echelon form

$$\begin{bmatrix} a & b & d & f & i & m & r \\ 0 & c & e & g & j & n & s \\ 0 & 0 & 0 & h & k & o & t \\ 0 & 0 & 0 & 0 & l & p & u \\ 0 & 0 & 0 & 0 & 0 & q & v \end{bmatrix}$$

What is an augmented matrix?

- augmented matrix

$$\left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right]$$

Homework #1-1

- Prove that by applying the “Gauss-Jordon Elimination” algorithm to an invertable matrix A, you can obtain A^{-1} , which is shown on the right-hand side of the final augmented matrix.

Why do we want to factorize A=LU ?

- Because it is easy to determine the inverse and determinant of the elimination matrix and the permutation matrix
- $A\mathbf{x}=\mathbf{b} \rightarrow L^{-1}LU\mathbf{x}=L^{-1}\mathbf{b} \rightarrow U\mathbf{x}=\mathbf{c}$
  - Where $U\mathbf{x}=\mathbf{c}$ is very easy to solve (Chap. 2 & Chap. 3)
- A=LU then
  - det(A)=det(L)det(U)=det(U)=對角線的連乘積 (Chap. 5)
- $A_1=L_1R_1 \rightarrow A_2=R_1L_1=L_2R_2 \rightarrow \ldots$ a matrix with decreasing eigen values on its diagonal ! (Chap. 6)

Lecture 1
Part B
Transpose, Inverse, & special matrices

Transpose and Inverse

- Definitions
  - A^T (transpose of matrix A) : $A_{ij} = (A^T)_{ji}$
  - A^{-1} (inverse matrix of a square matrix A) : $AA^{-1}=A^{-1}A=\mathbf{1}$
- 口訣：先穿襪子，再穿鞋子。回復時～
  先脫鞋子，再脫襪子。

$(AB)^T=B^TA^T$ $(AB)^{-1}=B^{-1}A^{-1}$

$(AB)^{-1}=B^{-1}A^{-1}$

- 證明：

$(AB)^{-1}(AB)=\mathbf{1}$

(1)

$B^{-1}A^{-1}(AB)=B^{-1}(A^{-1}A)B=B^{-1}(\mathbf{1})B=B^{-1}B=\mathbf{1}$

(2)

Equations (1) and (2) yield

$(AB)^{-1}=B^{-1}A^{-1}$

$(AB)^{-1}=B^{-1}A^{-1}$

- Example:

Let $A=\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad B=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Determine A^{-1}, B^{-1} ,$C=AB$, and $D=B^{-1}A^{-1}$

Verify your results by checking if they satisfy the following relations

$AA^{-1}=\mathbf{1}$, $BB^{-1}=\mathbf{1}$, and $CD=\mathbf{1}$

$(AB)^T = B^T A^T$

有兩種看法去驗證這個關係式。

第一種看法：

Let us consider two matrices $A_{m\times n}$ and $B_{n\times l}$

We have

$$A_{m\times n}B_{n\times l} = C_{m\times l}\ ,\quad (A^T)_{n\times m}\ ,\quad (B^T)_{l\times n}\ ,\quad (C^T)_{l\times m}$$

and $(B^T)_{l\times n}(A^T)_{n\times m} = (B^T A^T)_{l\times m}$

$\Rightarrow$ Both $(AB)^T = (C^T)$ and $(B^T A^T)$ are $l\times m$ matrices

$(AB)^T = B^T A^T$

第二種看法：

Let us consider the linear equations

$$\begin{aligned} 2x - 3y &= -4 \\ -x + 2y &= 3 \end{aligned}$$

They can be rewritten in the following matrix forms. Thus, we conclude that $(AB)^T = B^T A^T$.

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \quad \text{or} \quad A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \end{bmatrix} \quad \text{or} \quad \mathbf{x}^T A^T = \mathbf{b}^T$$

$(AB)^T = B^T A^T$ (比較嚴謹的證明)

Let us consider two matrices $A_{m\times n}$ and $B_{n\times l}$

For all $i \in \{1,....,m\}$ and $j \in \{1,....,l\}$, we have

$$\left((AB)^T\right)_{ji} = (AB)_{ij}$$ Definition of the transpose of a matrix

$$= \sum_{k=1}^{n} A_{ik} B_{kj}$$ Definition of matrix multiplication

$$= \sum_{k=1}^{n} (A^T)_{ki} (B^T)_{jk}$$ Definition of the transpose of a matrix

$$= \sum_{k=1}^{n} (B^T)_{jk} (A^T)_{ki}$$ Exchange of scalar multiplication

$$= (B^T A^T)_{ji}$$ Definition of matrix multiplication

注意,
這裡的
足標與
上課時
所用略
有不同

因為上式對所有的 *ji* 分量都成立, 故得證 $(AB)^T = B^T A^T$

Symmetric Matrix
&
Anti-symmetric Matrix

- Definitions
  - M is a symmetric matrix if $M^T = M$
  - M is an anti-symmetric matrix if $M^T = -M$

Decomposing a Square Matrix M to M^S+M^A

If M is a square matrix then M can be decomposed into a symmetric matrix M^S and an anti-symmetric matrix M^A

$M=M^S+M^A$

where

$M^S= (M + M^T)/2$

$M^A= (M – M^T)/2$

How to build a symmetric matrix (method 2)

If R is an m x n rectangular matrix then

- R^TR is an n x n symmetric matrix
- RR^T is an m x m symmetric matrix

  Proof:

  $(R^TR)^T=(R)^T(R^T)^T=R^TR$ ➔ R^TR is symmetric

  $(RR^T)^T=(R^T)^T(R)^T=RR^T$ ➔ RR^T is symmetric

- LDL^T is also a symmetric matrix

Symmetric Matrix

If M is a symmetric matrix, then

- M^{-1} is a symmetric matrix
- $M=LDL^T$ and $M=Q\Lambda Q^T$
  - The number of positive pivots in D and positive eigenvalues in Λ is the same.
  - A has real eigenvalues and orthonormal eigen vectors in Q 以後教

Expressions and Definitions

- Expressions of vector
  - Hand writing: $\vec{v}$ or $\underline{v}$ or |v> **or** $\vec{v}^T$ or $\underline{v}^T$ or <v|
  - Typeface: boldface for vector, i.e., $\mathbf{v}$ **or** $\mathbf{v}^T$
  - Matrix expression of vectors: $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ **or** $[\,v_1 \quad v_2 \quad v_3\,]$

where the vector is defined by

$$\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3 = \sum_i v_i\hat{e}_i = v_i\hat{e}_i$$

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 = \sum_i v_i\mathbf{e}_i = v_i\mathbf{e}_i$$

Vector Products

- Definitions of vector products
  - inner product: $\mathbf{v} \cdot \mathbf{w}$
    - the multiplying of the projection of one vector to the other vector
  - cross product: $\mathbf{v} \times \mathbf{w}$
    - size and normal direction of the area determined by the two vectors
  - dyad product: $\mathbf{vw}$

Matrix Expression of Vector Products

- inner product:

$$\mathbf{v}\cdot\mathbf{w} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

- cross product:

$$\mathbf{v}\times\mathbf{w} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

- dyad product:

$$\mathbf{vw} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} v_1w_1 & v_1w_2 & v_1w_3 \\ v_2w_1 & v_2w_2 & v_2w_3 \\ v_3w_1 & v_3w_2 & v_3w_3 \end{bmatrix}$$

Special Matrices (1)

- Definitions
  - A is a symmetric matrix if $A^T = A$
  - A is a anti-symmetric matrix if $A^T = -A$
  - A is an orthogonal matrix if $A^T = A^{-1}$ 以後教
  - Define z*: if z=a+ib then z*=a-ib
  - A is a Hermitian matrix if $A^H=(A^T)^*=A$ 以後教
  - A is a unitary matrix if $A^H=(A^T)^*=A^{-1}$ 以後教

Special Matrices (2)

- Identity matrix: **1** 教過了！
- Elimination matrix: 教過了！
- Permutation matrix: 教過了！
- Projection matrix: $P=P^2=P^T$ 以後教
- Rotation matrix: 以後教
- Reflection matrix: $Q=\mathbf{1}-2\mathbf{u}\mathbf{u}^T$ 以後教
- Householder matrix: $Q^T=Q^{-1}=Q$ 以後教

Row Operation,
Column Operation,
&
Matrices Multiplication

Column Operation

Solve $\mathbf{x}^T A^T = \mathbf{b}^T$ or $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \end{bmatrix}$

choose the Column eliminator on the right to be $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} b_1 & -2b_1 + b_2 \end{bmatrix}$$

Row Operation

To solve $\mathbf{x}^T A^T = \mathbf{b}^T$ or $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \end{bmatrix}$

is equivalent to solve $A\mathbf{x} = \mathbf{b}$ or $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ -2b_1 + b_2 \end{bmatrix}$$

Matrices multiplication

可利用

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}\begin{bmatrix} w & x \end{bmatrix} + \begin{bmatrix} b \\ d \end{bmatrix}\begin{bmatrix} y & z \end{bmatrix}$$

證明

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

Matrices multiplication

$$\begin{bmatrix}1 & 2\\3 & 2\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 4\\3 & 8\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}1 & 2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}=\begin{bmatrix}1\cdot[1 \quad 2]\\3\cdot[1 \quad 2]\end{bmatrix}+\begin{bmatrix}2\cdot[0 \quad 1]\\2\cdot[0 \quad 1]\end{bmatrix}$$

$$=\begin{bmatrix}1\cdot[1 \quad 2]+2\cdot[0 \quad 1]\\3\cdot[1 \quad 2]+2\cdot[0 \quad 1]\end{bmatrix}$$ i.e., $$\begin{bmatrix}1 & 2\\3 & 2\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 4\\3 & 8\end{bmatrix}$$ yields row operation

$$\Rightarrow 1\cdot[1 \quad 2]+2\cdot[0 \quad 1]=[1 \quad 2]\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}=[1 \quad 4]$$

$$\& \quad 3\cdot[1 \quad 2]+2\cdot[0 \quad 1]=[3 \quad 2]\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}=[3 \quad 8]$$

Put the Row Operator on the left! e.g., row elimination, or row permutation

Matrices multiplication

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} & 2 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 0 \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} & 1 \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 0 \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} & 2 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} \end{bmatrix}$$

i.e., $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 8 \end{bmatrix} \Rightarrow 1 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 0 \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

& $2 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

Put the Column Operator on the right!
e.g., column elimination,
or column permutation