# Lecture 5 Drift motion: Multiple-time-scale analysis

Key points

$\boldsymbol{E}\times\boldsymbol{B}$ Drift

Gravitational Drift

- **Application:** Gravitational Rayleigh-Taylor Instability => plasma bubble in the E-region ionosphere

Polarization Drift

- **Applications:** Polarization current at the wave front of the MHD waves

Curvature Drift

Grad B Drift

- **Application:** Formation of partial ring current after substorm injections

Review: 飄（漂）移運動 drift motion

帶電粒子，除了會繞著磁場打轉 gyro motion，也會在兩個強磁場所構成的磁瓶中進行彈跳動 bounce motion。最後還會在不均勻的磁場結構中進行飄移運動 drift motion。在 drift motion 過程中，平均一個迴旋週期的時間裡，粒子的平均位置為 guiding center， 也就是說，帶電粒子本身會繞著 guiding center 打轉。而 guiding center 會發生各種飄移。

最常見的一種飄移運動，就是 $\boldsymbol{E}\times\boldsymbol{B}$ drift。其他還有

gravitational drift
polarization drift
grad B drift
curvature drift

至少有以下四種方式，來了解 $\boldsymbol{E}\times\boldsymbol{B}$ drift

1. Solving the equation of motion with the given E field and B field directly.
2. Changing the moving frame to the guiding canter moving frame, where E=0, then changing back to the original moving frame with a DC Electric field.
3. Considering the change of kinetic energy, thus the change of gyro radius.
4. Taking time-averaging to remove the high-frequency part of the motion.

第 4 種方法， 可適用於所有 multiple-timescale processes.

尤其當兩個 timescales 的時間尺度相差很多的時候。

多時間尺度問題標準處理法：

Step 1. 先找出最高頻方程式。

Step 2. 求最高頻現象的解。

Step 3. 將原來方程式對時間平均，消去最高頻物理現象，留下一個包含次高頻時間尺度的方程式。

Step 4. 求次高頻現象的解。

Step 5. 將原來方程式對時間平均，消去次高頻物理現象。

Step 6. 留下下一個時間尺度的方程式。

Step 7. 於此類推，直到 DC 現象（steady state 現象）出現為止。

## $\boldsymbol{E}\times\boldsymbol{B}$ Drift

命題：

Let us consider a charge particle with mass $m$, and charge $q$, moving in a space with uniform and steady magnetic field $\vec{B}$ and electric field $\vec{E}$. Assume that the electric field is perpendicular to the magnetic field $(\vec{E} \perp \vec{B})$ and the initial velocity of the particle is also perpendicular to the ambient magnetic field, $(\vec{v}(t=0) \perp \vec{B})$.

基本方程：

The equation of motion of the charge particle is

$$m\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v}\times\vec{B}) \tag{5.1}$$

以均勻電磁場中的 $\boldsymbol{E}\times\boldsymbol{B}$ **Drift** 這個問題而言，只有兩個時間尺度：高頻現象與 steady state 現象。

The velocity of the charge particle can be decomposed into two components, a high-frequency component $\vec{v}^H$ and a low-frequency component $\vec{v}^L$. That is

| | |
|---|---|
| $\vec{v} = \vec{v}^H + \vec{v}^L$ | (5.2) |

Substituting Eq. (5.2) into Eq. (5.1) to eliminate $\vec{v}$, it yields

| | |
|---|---|
| $m(\frac{d\vec{v}^H}{dt} + \frac{d\vec{v}^L}{dt}) = q(\vec{E} + \vec{v}^H\times\vec{B} + \vec{v}^L\times\vec{B})$ | (5.3) |

Step 1. 先找出最高頻方程式：

The high-frequency components in Eq. (5.3) yields

$$m\left(\frac{d\vec{v}^H}{dt}\right) = q(\vec{v}^H \times \vec{B}) \tag{5.4}$$

Thus, the high-frequency velocity component is the gyro velocity component ($\vec{v}^H = \vec{v}^H_{gyro}$). Solution of Eq. (5.4) has been discussed in Lecture 2 (Example 2.1).

Step 2. 求最高頻現象的解：

For $\Omega_c = |q|B/m$, the solution of Eq. (5.4) is

$$\vec{v}^H(t) = v_\perp[\hat{e}_{\perp 1}\sin(\Omega_c t + \phi) + \frac{q}{|q|}\hat{e}_{\perp 2}\cos(\Omega_c t + \phi)] \tag{5.5}$$

where $\phi$ is the initial phase angle and $\hat{e}_{\perp 1} \times \hat{e}_{\perp 2} = \hat{e}_\parallel = \hat{B}$

Step 3. 將原來方程式對時間平均，消去最高頻物理現象，留下一個包含次高頻時間尺度的方程式。

Let us define the time averaging of A by

$$< A >_{2\pi/\Omega_c} = \frac{\int_0^{\frac{2\pi}{\Omega_c}} A(t)dt}{\int_0^{\frac{2\pi}{\Omega_c}} dt} \tag{5.6}$$

The low-frequency equation of motion can be obtained by time-averaging of Eq. (5.3) over one gyro period. Since

$$\int_0^{\frac{2\pi}{\Omega_c}} \sin(\Omega_c t + \phi)dt = \int_0^{\frac{2\pi}{\Omega_c}} \cos(\Omega_c t + \phi)dt = 0$$

it yields

$$< \vec{v}^H >_{2\pi/\Omega_c} = 0$$

$$< \frac{d\vec{v}^H}{dt} >_{2\pi/\Omega_c} = 0$$

Thus, time-averaging of Eq. (5.3) over one gyro period yields

| | |
|---|---|
| $m(< \frac{d\vec{v}^L}{dt} >_{2\pi/\Omega_c}) = q(\vec{E} + < \vec{v}^L >_{2\pi/\Omega_c} \times \vec{B})$ | (5.7) |

Let us assume that $\vec{v}^L$ is constant with time. That is $d\vec{v}^L/dt = 0$ and $< \vec{v}^L >_{2\pi/\Omega_c} = \vec{v}^L$.（這個假設，等答案求出來後，需要被檢驗。） For steady $\vec{v}^L$, Eq. (5.7) yields

| | |
|---|---|
| $0 = q(\vec{E} + \vec{v}^L \times \vec{B})$ | (5.8) |

Step 4-7. 求次高頻現象的解，在此也是 steady state 的解

Solution of Eq. (5.8) is

$$\vec{v}^L = \frac{\vec{E}\times\vec{B}}{B^2} \tag{5.9}$$

Namely, $|\vec{v}^L| = E/B$ and $\hat{v}^L = \hat{E}\times\hat{B}$.

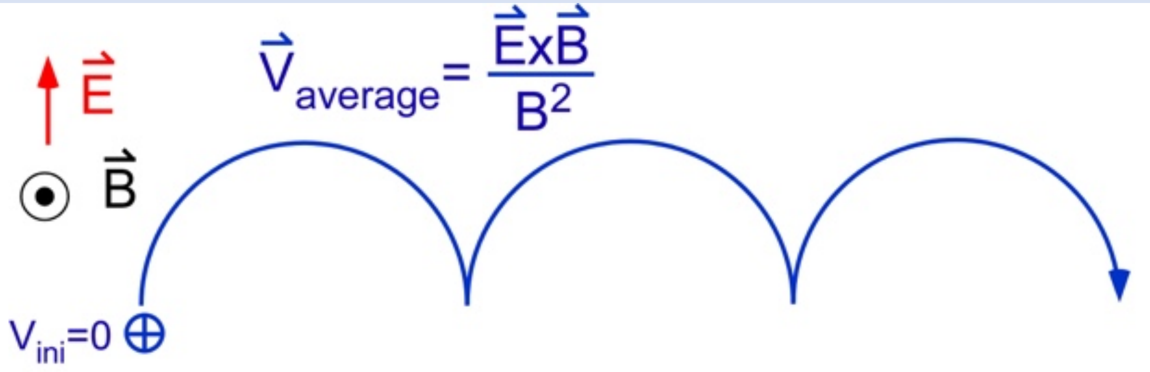
檢驗答案

- One can substitute the solution given in Eq. (5.9) back to Eq. (5.8) and find that the solution satisfies the Eq. (5.8).
- The solution given in Eq. (5.9) also satisfies the time independent assumption $d\vec{v}^L/dt = 0$.

The low-frequency velocity component given in Eq. (5.9) is called the "$\boldsymbol{E}\times\boldsymbol{B}$ drift velocity" (e.g., Fig. 5.1).

It is important to note that, since we consider non-relativistic equation of motion, the $\boldsymbol{E}\times\boldsymbol{B}$ drift speed $E/B$ must satisfies $E/B \ll c$. Namely, for a given uniform electric field $\vec{E}$, the background magnetic field strength should be much greater than $E/c$. Or, for a given uniform magnetic field $\vec{B}$, the background electric field strength should be much less than $cB$.

## $E\times B$ Drift




Choose $V_{E\times B}=1$

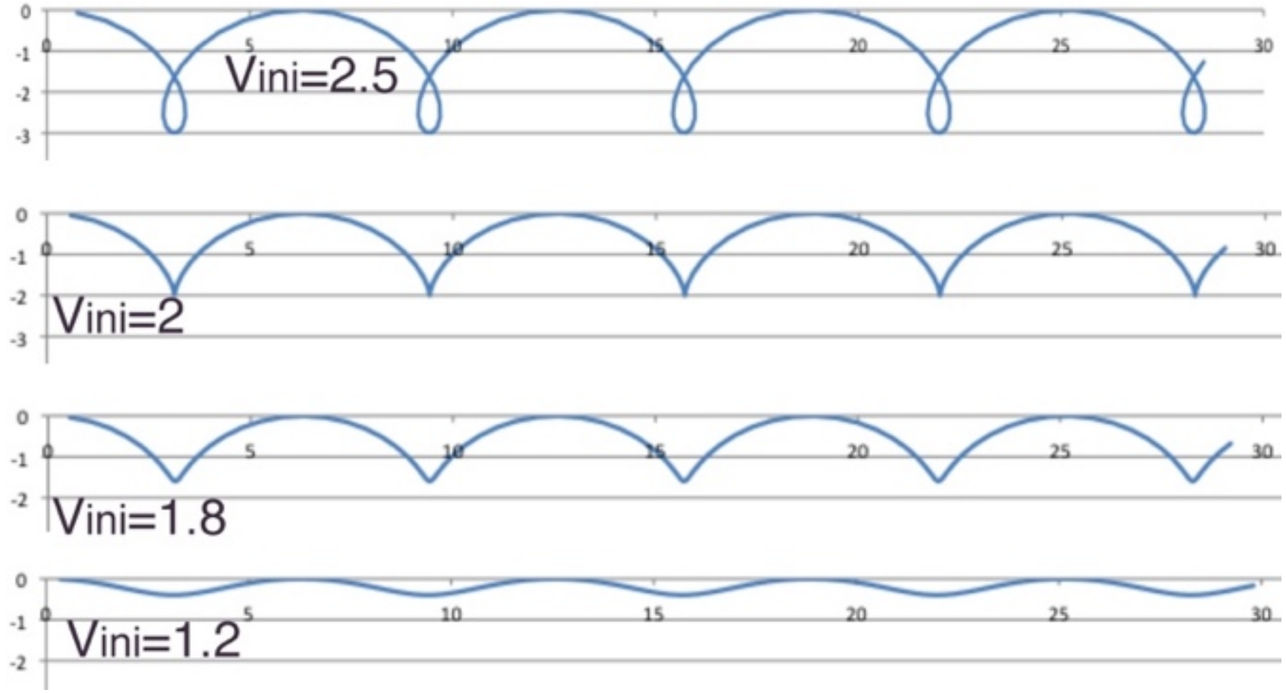



Fig. 5.1. Examples of $E\times B$ Drift

**Gravitational drift**

Let us consider a charge particle with mass $m$, and charge $q$, moving in a uniform magnetic field $\vec{B}$ and a uniform gravitational field $\vec{g}$. Assume that $\vec{g} \perp \vec{B}$ and the initial velocity of the particle is also perpendicular to the ambient magnetic field, i.e., $\vec{v}(t=0) \perp \vec{B}$.

The equation of motion of the charge particle is

$$m\frac{d\vec{v}}{dt} = m\vec{g} + q\vec{v}\times\vec{B} \tag{5.10}$$

The velocity of the charge particle can be decomposed into two components, a high-frequency component $\vec{v}^H$ and a low frequency component $\vec{v}^L$. That is

$$\vec{v} = \vec{v}^H + \vec{v}^L \tag{5.11}$$

Substituting Eq. (5.11) into Eq. (5.10), it yields

$$m(\frac{d\vec{v}^H}{dt} + \frac{d\vec{v}^L}{dt}) = m\vec{g} + q(\vec{v}^H \times \vec{B} + \vec{v}^L \times \vec{B}) \quad (5.12)$$

The high-frequency components in Eq. (5.12) yields

$$m(\frac{d\vec{v}^H}{dt}) = q(\vec{v}^H \times \vec{B}) \quad (5.13)$$

Again, the high-frequency velocity component is the gyro velocity component. Time-averaging of Eq. (5.12) yields

$$m(< \frac{d\vec{v}^L}{dt} >_{2\pi/\Omega_c}) = m\vec{g} + q < \vec{v}^L >_{2\pi/\Omega_c} \times \vec{B} \quad (5.14)$$

Assume that $\vec{v}^L$ is constant with time. That is $d\vec{v}^L(t)/dt = 0$ and $< \vec{v}^L >_{2\pi/\Omega_c} = \vec{v}^L$. Eq. (5.14) yields

$$0 = m\vec{g} + q\vec{v}^L \times \vec{B} \quad (5.15)$$

Solution of Eq. (5.15) is

$$\vec{v}^L = \frac{m\vec{g}\times\vec{B}}{qB^2} \tag{5.16}$$

One can substitute the solution in Eq. (5.16) back to Eq. (5.15) and find that the solution satisfies the Eq. (5.15). The solution given in Eq. (5.16) also satisfies the time independent condition $d\vec{v}^L/dt = 0$. The low-frequency velocity component given in Eq. (5.16) is called the "gravitational drift velocity." Note that the gravitational drift velocities of ions and electrons are in different directions. The gravitational drift speed increases with increasing mass of the charge particle.

## General form of the low-frequency drift velocity

Let us consider a charge particle with mass $m$, and charge $q$, moving in a uniform magnetic field $\vec{B}$ and a uniform Force field $\vec{F}$. i.e.,

$$m\frac{d\vec{v}}{dt} = \vec{F} + q\vec{v}\times\vec{B} \quad (5.17)$$

The solution of Eq. (5.17) can be written as $\vec{v} = \vec{v}^{H}_{gyro} + \vec{v}^{L}_{drift}$. The low-frequency drift velocity can be written as

$$\vec{v}^{L}_{drift} = \frac{\vec{F}\times\vec{B}}{qB^2} \quad (5.18)$$

**Application:** Gravitational-Rayleigh-Taylor Instability -- An example of mixed $\boldsymbol{E}\times\boldsymbol{B}$ drift and gravitational drift

Either due to the sunset disturbance or due to a gravity wave disturbance, the surface disturbance on the bottom side of the ionosphere (the D-region and E-region ionosphere) is unstable to the Gravitational-Rayleigh-Taylor Instability.

**Q:** What is gravity wave （重力波）?

**Ans.:** 在一個對流穩定的大氣中，火山爆發，或颱風，或過山風所造成的上下擾動加上水平風，就可以產生重力波。

**Exercise 5.1.** Please discuss the process of the Gravitational-Rayleigh-Taylor Instability.

**Polarization drift**

Let us consider a charge particle with mass $m$, and charge $q$, moving in a uniform magnetic field $\vec{B} = \hat{z}B$ and a slowly increased electric field $\vec{E}(t) = \hat{y}\dot{E}t.$, where $\dot{E}$ is constant with time. Assume that the initial velocity of the particle is perpendicular to the ambient magnetic field.

The equation of motion of the charge particle is

$$m\frac{d\vec{v}(t)}{dt} = q[\vec{E}(t) + \vec{v}(t)\times\vec{B}] \quad (5.19)$$

We decompose the velocity of the charge particle into three components, a high-frequency component $\vec{v}_{gyro}^{H}(t)$, a slowly changed component $\vec{v}_{E\times B}^{L}(t)$, and a DC drift component $\vec{v}_{drift}^{DC}$, i.e.,

| | |
|---|---|
| $\vec{v}(t) = \vec{v}_{gyro}^{H}(t) + \vec{v}_{E\times B}^{L}(t) + \vec{v}_{drift}^{DC}$ | (5.20) |

Substituting Eq. (5.20) into Eq. (5.19), it yields

| | |
|---|---|
| $m(\frac{d\vec{v}_{gyro}^{H}}{dt} + \frac{d\vec{v}_{E\times B}^{L}}{dt}) = q(\vec{E} + \vec{v}_{gyro}^{H}\times\vec{B} + \vec{v}_{E\times B}^{L}\times\vec{B} + \vec{v}_{drift}^{DC}\times\vec{B})$ | (5.21) |

where the high-frequency component $\vec{v}_{gyro}^{H}$ satisfies

| | |
|---|---|
| $m(\frac{d\vec{v}_{gyro}^{H}}{dt}) = q(\vec{v}_{gyro}^{H}\times\vec{B})$ | (5.22) |

and the slowly changed velocity component satisfies

| | |
|---|---|
| $\vec{E}(t) + \vec{v}_{E\times B}^{L}(t)\times\vec{B} = 0$ | (5.23) |
| $\vec{v}_{E\times B}^{L}(t) = \frac{\vec{E}(t)\times\vec{B}}{B^2}$ | (5.24) |

$$\frac{d\vec{v}^L_{E\times B}(t)}{dt} = \frac{1}{B^2}\frac{d\vec{E}(t)}{dt}\times\vec{B} = \frac{\hat{y}\dot{E}}{B^2}\times\vec{B} \qquad (5.25)$$

The equation of motion at lower frequency can be obtained by time-averaging of Eq. (5.21) over one gyro period. Since

$$< \vec{v}^H >_{2\pi/\Omega_c} = 0$$

$$< \frac{d\vec{v}^H}{dt} >_{2\pi/\Omega_c} = 0$$

$$< \frac{d\vec{v}^L_{E\times B}(t)}{dt} >_{2\pi/\Omega_c} = < \frac{\hat{y}\dot{E}}{B^2}\times\vec{B} >_{2\pi/\Omega_c} = \frac{\hat{y}\dot{E}}{B^2}\times\vec{B}$$

the time-averaging of Eq. (5.21) can be written as

$$m\frac{\hat{y}\dot{E}}{B^2}\times\vec{B} = q(< \vec{E} + \vec{v}^L_{E\times B}(t)\times\vec{B} >_{\frac{2\pi}{\Omega_c}} + \vec{v}^{DC}_{drift}\times\vec{B}) \qquad (5.26)$$

Substituting Eq. (5.23) into Eq. (5.26) to eliminate the term $\vec{E} + \vec{v}^{L}_{E\times B}(t)\times\vec{B}$ in Eq. (5.26), it yields

$$m\frac{\hat{y}\dot{E}}{B^2}\times\vec{B} = q\vec{v}^{DC}_{drift}\times\vec{B} \qquad (5.27)$$

Eq. (5.27) yields

$$\vec{v}^{DC}_{drift} = \hat{y}\dot{E}\frac{m}{qB^2} \qquad (5.28)$$

The drift velocity shown in Eq. (5.28) is called the "polarization drift velocity," because it is associated with the polarization electric field at the wave front of a low-frequency MHD (magnetohydrodynamic) waves. The electric current resulting from the polarization drifts of ions and electrons is called the "polarization current."

According to Eq. (5.28), the polarization current at the wave front of the MHD wave should be parallel to the $d\vec{E}(t)/dt$ (e.g., Fig. 5.2).

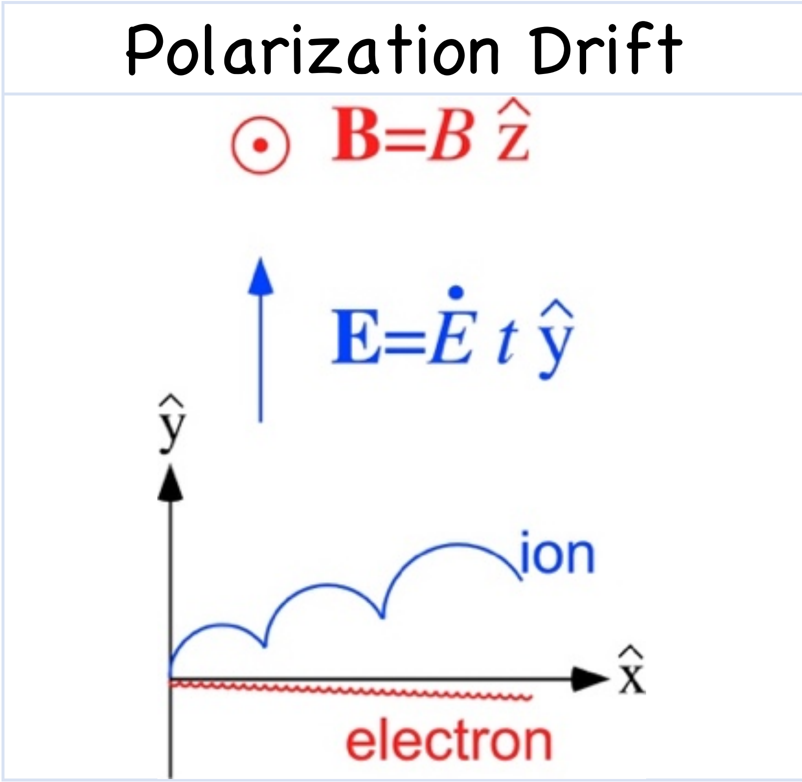



Fig. 5.2. A sketch of polarization drifts of an ion and an electron.

**Curvature drift**

Let us consider a charge particle with mass $m$, and charge $q$, moving in a steady but curved magnetic field $\vec{B}$. Let $R_B$ be the radius of curvature of the magnetic field line. If the charge particle has a velocity component $v_{\parallel}$ parallel to the local magnetic field, then in the charge particle's guiding center moving frame, the particle will sense a centrifugal force $\vec{F}_c$, where

$$\vec{F}_c = \hat{R}_B \frac{m v_{\parallel}^2}{R_B} \quad (5.29)$$

Substituting Eq. (5.29) into the general form of the low-frequency drift velocity Eq. (5.18), it yields

$$\vec{v}^{L}_{drift} = \frac{mv_{\parallel}^{2}}{R_B}\frac{\hat{R}_B \times \vec{B}}{qB^2} \tag{5.30}$$

The drift velocity given in Eq. (5.30) is called the "curvature drift velocity" (e.g., Fig. 5.3).

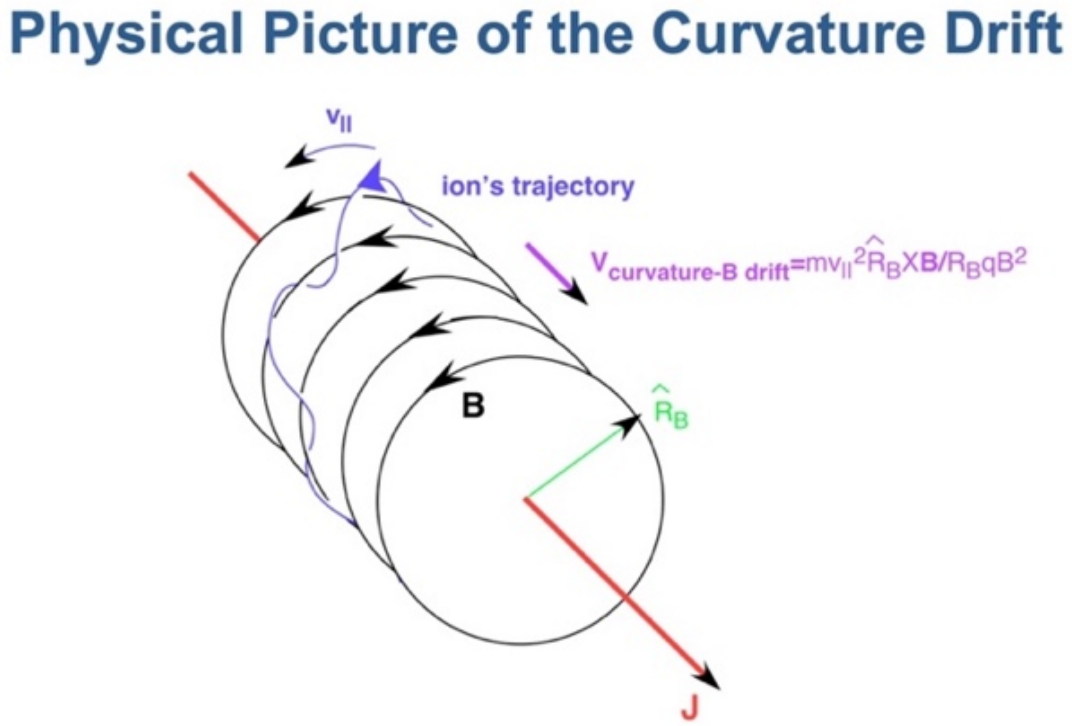



Fig. 5.3. A sketch of curvature drift of an ion.

**Q:** How to determine the radius of curvature of a given magnetic field $\vec{B}$ ?

**Ans.:** Considering a space curve, let $\hat{t}$ be the unit tangent vector along the curve and $s$ denote the length along the curve. Since

$$\frac{d\hat{t}}{ds} = \lim_{\Delta s \to 0} \frac{\hat{t}(s+\Delta s) - \hat{t}(s)}{\Delta s} = \lim_{\Delta\theta \to 0} \frac{|\hat{t}|\Delta\theta}{r\Delta\theta}(-\hat{n}) = \frac{-\hat{n}}{r} \qquad (5.31)$$

we have

$$\frac{\hat{n}}{r} = -\frac{d\hat{t}}{ds} = -\hat{t}\cdot\nabla\hat{t} \qquad (5.32)$$

where $\hat{n}$ is the unit vector along the radial direction of the curve, and $r$ is the radius of curvature.

If we consider the curvature of a magnetic field line, the $\hat{n}$ becomes $\hat{R}_B$, $r$ is the $R_B$, $\hat{t}$ is along the magnetic field direction $\hat{B}$. Thus, we have

$$\frac{\hat{R}_B}{R_B} = -\hat{B} \cdot \nabla \hat{B} \qquad (5.33)$$

For $\hat{B} = \vec{B}/B$, it yields

$$\frac{\hat{R}_B}{R_B} = -\frac{\vec{B} \cdot \nabla \vec{B}}{B^2} + \frac{\hat{B}\hat{B}}{B} \cdot \nabla B = -\frac{\vec{B} \cdot \nabla \vec{B}}{B^2} + \frac{\nabla_{\parallel} B}{B} \qquad (5.34)$$

Note that, if we ignore the displacement current, the $\boldsymbol{J}\times\boldsymbol{B}$ Lorentz force can be decomposed into a magnetic pressure gradient force and a magnetic tension force. That is

$$\vec{J}\times\vec{B} \approx \frac{\nabla\times\vec{B}}{\mu_0}\times B = \frac{\vec{B}\cdot\nabla\vec{B}}{\mu_0} - \frac{\nabla B^2}{2\mu_0} = -\frac{\nabla_\perp B^2}{2\mu_0} - \frac{\nabla_\parallel B^2}{2\mu_0} + \frac{\vec{B}\cdot\nabla\vec{B}}{\mu_0} \quad (5.35)$$

where $\nabla_\parallel = \hat{B}\hat{B}\cdot\nabla$ and $\nabla_\perp = \nabla - \nabla_\parallel = (\overleftrightarrow{1} - \hat{B}\hat{B})\cdot\nabla$ .

Substituting Eq. (5.34) into Eq. (5.35), it yields

$$\vec{J}\times\vec{B} = -\nabla_\perp\left(\frac{B^2}{2\mu_0}\right) - \frac{\hat{R}_B}{R_B}\frac{B^2}{\mu_0} \quad (5.36)$$

magnetic pressure gradient force

magnetic tension force

Substituting Eq. (5.36) into Eq. (5.31) to eliminate $\hat{R}_B/R_B$, the curvature drift velocity given in Eq. (5.31) can be rewritten as

$$\vec{v}^L_{curv.} = \frac{mv_\parallel^2}{qB^2}\left(\frac{\mu_0}{B^2}\right)\left[-\nabla_\perp\left(\frac{B^2}{2\mu_0}\right) - \vec{J}\times\vec{B}\right]\times\vec{B}$$
$$= \frac{mv_\parallel^2}{qB^3}\left(-\nabla_\perp B\times\vec{B}\right) - \frac{mv_\parallel^2}{qB^2}\left(\frac{\mu_0}{B^2}\right)(\vec{J}\times\vec{B})\times\vec{B} \quad (5.37)$$

Note that, for $\vec{J} = 0$, the curvature drift velocity becomes

$$\vec{v}^L_{curv.} = \frac{mv_\parallel^2}{qB^3}\left(-\nabla_\perp B\times\vec{B}\right) \quad (5.38)$$

The resulting equation is "similar" to the grad B drift

$$\vec{v}^L_{\nabla B} = \frac{mv_\perp^2}{2qB^3}\left(-\nabla_\perp B\times\vec{B}\right) \quad (5.39)$$

**Grad B drift**

Let us consider a charge particle with mass $m$, and charge $q$, moving in a non-uniform magnetic field $\vec{B} = \hat{z}B(x)$. Assume that the initial velocity of the particle is perpendicular to the ambient magnetic field. The equation of motion of the charge particle is

| | |
|---|---|
| $$\frac{d\vec{r}}{dt} = \vec{v}$$ | (5.39) |
| $$m\frac{d\vec{v}}{dt} = q\vec{v}\times\vec{B}$$ | (5.40) |

The velocity of the charge particle can be decomposed into a high-frequency gyro motion component and a low-frequency guiding center drift component. That is,

| | |
|---|---|
| $\vec{r} = \vec{r}^{H}_{gyro} + \vec{r}^{L}_{g.c.}$ | (5.41) |
| $\vec{v} = \vec{v}^{H}_{gyro} + \vec{v}^{L}_{g.c.}$ | (5.42) |

where $\vec{r}^{H}_{gyro}$ and $\vec{v}^{H}_{gyro}$ satisfy the following high-frequency equations

| | |
|---|---|
| $\frac{d\vec{r}^{H}_{gyro}}{dt} = \vec{v}^{H}_{gyro}$ | (5.43) |
| $m\frac{d\vec{v}^{H}_{gyro}}{dt} = q\vec{v}^{H}_{gyro}\times\vec{B}(\vec{r}^{L}_{g.c.})$ | (5.44) |

The solutions of $\vec{v}^{H}_{gyro}(t)$ and $\vec{r}^{H}_{gyro}(t)$ can be written as

| | |
|---|---|
| $\vec{v}^{H}_{gyro}(t) = v_{\perp}[\hat{x}\sin(\Omega_c t + \phi) + \frac{q}{\lvert q\rvert}\hat{y}\cos(\Omega_c t + \phi)]$ | (5.45) |

$$\vec{r}^{H}_{gyro}(t) = \frac{v_\perp}{\Omega_c}\left[-\hat{x}\cos(\Omega_c t + \phi) + \frac{q}{|q|}\hat{y}\sin(\Omega_c t + \phi)\right] \quad (5.46)$$

where $\phi$ is the initial phase angle and $\Omega_c = |q|B(\vec{r}^{L}_{g.c.})/m$ is the gyro (angular) frequency.

The Taylor series expansion of the magnetic field with respect to the guiding center is

$$\vec{B}(\vec{r}) = \vec{B}\left(\vec{r}^{L}_{g.c.}\right) + \left(\vec{r} - \vec{r}^{L}_{g.c.}\right) \cdot \nabla\vec{B}\Big|_{\vec{r}=\vec{r}^{L}_{g.c.}} + \cdots \quad (5.47)$$

Substituting Eqs. (5.41), (5.42) & (5.47) into Eq. (5.40), it yields

$$m\frac{d\vec{v}^H_{gyro}}{dt} + m\frac{d\vec{v}^L_{g.c.}}{dt} = q\left(\vec{v}^H_{gyro} + \vec{v}^L_{g.c.}\right)\times\vec{B}\left(\vec{r}^L_{g.c.}\right) + q\left(\vec{v}^H_{gyro} + \vec{v}^L_{g.c.}\right)\times\left(\vec{r} - \vec{r}^L_{g.c.}\right)\cdot\nabla\vec{B}\Big|_{\vec{r}=\vec{r}^L_{g.c.}} + \cdots \quad (5.48)$$

Averaging Eq. (5.48) over one gyro period, ignoring the higher order terms, and assuming that $\vec{v}^L_{g.c.}$ is a constant with time, it yields the first-order approximation

$$0 = \vec{v}^L_{g.c.}\times\vec{B}\left(\vec{r}^L_{g.c.}\right) + < \vec{v}^H_{gyro}\times\vec{r}^H_{gyro}\cdot\nabla\vec{B}\Big|_{\vec{r}=\vec{r}^L_{g.c.}} >_{2\pi/\Omega_c} \quad (5.49)$$

Since $\vec{B} = \hat{z}B(x)$, it yields

$$\vec{r}^{H}_{gyro} \cdot \nabla\vec{B}\Big|_{\vec{r}=\vec{r}^{L}_{g.c.}} = x^{H}_{gyro} \frac{dB}{dx}\Big|_{\vec{r}=\vec{r}^{L}_{g.c.}} \hat{z} \quad (5.50)$$

Thus, we have

$$\vec{v}^{H}_{gyro} \times \vec{r}^{H}_{gyro} \cdot \nabla\vec{B}\Big|_{\vec{r}=\vec{r}^{L}_{g.c.}} = +\hat{x}{v_y}^{H}_{gyro} x^{H}_{gyro} \frac{dB}{dx}\Big|_{\vec{r}=\vec{r}^{L}_{g.c.}} - \hat{y}{v_x}^{H}_{gyro} x^{H}_{gyro} \frac{dB}{dx}\Big|_{\vec{r}=\vec{r}^{L}_{g.c.}} \quad (5.51)$$

Substituting Eqs. (5.45) & (5.46) into Eq.(5.51), it yields

$$\vec{v}_{gyro}^{H} \times \vec{r}_{gyro}^{H} \cdot \nabla \vec{B}\Big|_{\vec{r}=\vec{r}_{g.c.}^{L}}$$
$$= -\hat{x}\frac{q}{|q|}\frac{v_\perp^2}{\Omega_c}\cos(\Omega_c t+\phi)\cos(\Omega_c t+\phi)\frac{dB}{dx}\Big|_{\vec{r}=\vec{r}_{g.c.}^{L}}$$
$$+\hat{y}\frac{v_\perp^2}{\Omega_c}\sin(\Omega_c t+\phi)\cos(\Omega_c t+\phi)\frac{dB}{dx}\Big|_{\vec{r}=\vec{r}_{g.c.}^{L}} \qquad (5.52)$$

The time average of Eq. (5.52) becomes

$$< \vec{v}_{gyro}^{H} \times \vec{r}_{gyro}^{H} \cdot \nabla \vec{B}\Big|_{\vec{r}=\vec{r}_{g.c.}^{L}} >_{\frac{2\pi}{\Omega_c}} = -\hat{x}\frac{q}{|q|}\frac{v_\perp^2}{2\Omega_c}\frac{dB}{dx}\Big|_{\vec{r}=\vec{r}_{g.c.}^{L}} \qquad (5.53)$$

where $< \sin(\Omega_c t+\phi)\cos(\Omega_c t+\phi) >_{2\pi/\Omega_c} = 0$ and

$$< \cos(\Omega_c t+\phi)\cos(\Omega_c t+\phi) >_{2\pi/\Omega_c} = \frac{1}{2}$$

For $\Omega_c = |q|B(\vec{r}^L_{g.c.})/m$, Eq. (5.53) can be rewritten as

$$< \vec{v}^H_{gyro} \times \vec{r}^H_{gyro} \cdot \nabla\vec{B}\Big|_{\vec{r}=\vec{r}^L_{g.c.}} >_{\frac{2\pi}{\Omega_c}} = -\frac{mv_\perp^2}{2qB(\vec{r}^L_{g.c.})}\nabla B|_{\vec{r}=\vec{r}^L_{g.c.}} \qquad (5.54)$$

Substituting Eq. (5.54) into Eq. (5.49), it yields

$$0 = \vec{v}^L_{g.c.} \times \vec{B}(\vec{r}^L_{g.c.}) - \frac{mv_\perp^2}{2qB(\vec{r}^L_{g.c.})}\nabla B|_{\vec{r}=\vec{r}^L_{g.c.}} \qquad (5.55)$$

We can obtain $\vec{v}^L_{g.c.}$ from Eq. (5.55). That is

$$\vec{v}^L_{g.c.} = \frac{mv_\perp^2}{2}\frac{-\nabla B|_{\vec{r}=\vec{r}^L_{g.c.}} \times \vec{B}(\vec{r}^L_{g.c.})}{qB^3|_{\vec{r}=\vec{r}^L_{g.c.}}} \qquad (5.56)$$

Note that, Eq. (5.56) is only a first-order approximation of the grad B drift. The magnetic field and its gradient in Eq. (5.56) should be evaluated at the guiding center.

**Grad B drift 偷懶取巧的記憶法(物理意義錯誤)**

For constant magnetic momentum $\mu = W_\perp / B$,
it yields $W_\perp = \mu B$, and $-\nabla W_\perp = -\mu \nabla B$.
The dimension of $-\nabla W_\perp$ is a force.
It yields, the dimension of $-\mu \nabla B$ is also a force.
Substituting $\vec{F} = -\mu \nabla B$ into Eq. (5.18) yields

$$\vec{v}_{drift}^{L} = \frac{-\mu \nabla B \times \vec{B}}{qB^2} = \frac{W_\perp}{B} \frac{-\nabla B \times \vec{B}}{qB^2} = \frac{m v_\perp^2}{2} \frac{-\nabla B \times \vec{B}}{qB^3} \quad (5.57)$$

The biggest problem of this approach is that $-\nabla W_\perp$ is NOT a force. It did not show that this result is only a first-order approximation, nor the location where the magnetic field and its gradient should be evaluated.