Lecture 2 Gyromotion and Plasma Oscillation

Key points

Cyclotron frequency (gyrofrequency) of the αth species $f_{c\alpha} = \Omega_{c\alpha}/2\pi$

$$\Omega_{c\alpha} = eB/m_\alpha$$

Plasma frequency of the αth species

$$f_{p\alpha} = \omega_{p\alpha}/2\pi$$

$$\omega_{p\alpha} = (n_\alpha e^2/\epsilon_0 m_\alpha)^{1/2}$$

Other combinations:

upper hybrid frequency $f_{UH} = \omega_{UH}/2\pi$

$$\omega_{UH} = \sqrt{\omega_{pe}^2 + \Omega_{ce}^2}$$

lower hybrid frequency $f_{LH} = \omega_{LH}/2\pi$

$$\omega_{LH} = \sqrt{\Omega_{ce}\Omega_{ci}}$$

Example 2.1.

Let us consider an ion with charge e, mass m_i, moving in a space with *uniform* magnetic field $\vec{B} = \hat{z}B$. Let the initial position and velocity of the ion satisfy $\vec{r}(t=0) = 0$ and $\vec{v}(t=0) = \hat{x}v_0$. For $v_0 \ll c$, the ion's equations of motion can be written as

$$\frac{d\vec{r}(t)}{dt} = \vec{v}(t)$$

$$\frac{d\vec{v}(t)}{dt} = \frac{e}{m_i}\vec{v}(t)\times\vec{B}$$

Hint: 寫出分量後，再聯立求解.

$$\vec{r}(t) = \hat{x}x(t) + \hat{y}y(t) + \hat{z}z(t)$$

$$\vec{v}(t) = \hat{x}v_x(t) + \hat{y}v_y(t) + \hat{z}v_z(t)$$

求解以前，先畫一下正離子運動軌跡，估算一下解的形式。再求數值解，兩相驗證，結果一致，就應該是正確答案了。

第 1 步：在 x- y 平面上，繪出正離子軌跡，估計 $x(t)$ and $y(t)$ 的函數形式。

第 2 步：在 v_x- v_y 平面，繪出正離子在速度空間中的軌跡，估計$v_x(t)$ and $v_y(t)$ 的函數形式。

第 3 步：解以下聯立常微分方程式 Solving the following system ODEs

$$\dot{x}(t) = v_x(t) \tag{2.1}$$

$$\dot{y}(t) = v_y(t) \tag{2.2}$$

$$\dot{v}_x(t) = \frac{eB}{m_i} v_y(t) \tag{2.3}$$

$$\dot{v}_y(t) = -\frac{eB}{m_i} v_x(t) \tag{2.4}$$

Taking time deriviatives of Equation (2.3) and substituting Equation (2.4) into the resulting equation to eliminate $\dot{v}_y(t)$, it yields

$$\ddot{v}_x(t) = -\left(\frac{eB}{m_i}\right)^2 v_x(t) \tag{2.5}$$

Let $\Omega_{ci} = eB/m_i$. Solution of Equation (2.5) should be given in the following form

$$v_x(t) = C_1 \cos(\Omega_{ci} t) + C_2 \sin(\Omega_{ci} t) \quad (2.6)$$

Substituting Equation (2.6) into Equation (2.3), it yields

$$v_y(t) = C_2 \cos(\Omega_{ci} t) - C_1 \sin(\Omega_{ci} t) \quad (2.7)$$

Since $v_x(t=0) = v_0$ and $v_y(t=0) = 0$, it yields $C_1 = v_0$ and $C_2 = 0$. Thus, we have

$$v_x(t) = v_0 \cos(\Omega_{ci} t) \quad (2.8)$$

$$v_y(t) = -v_0 \sin(\Omega_{ci} t) \quad (2.9)$$

Substituting Equations (2.8) and (2.9) into Equations (2.1) and (2.2), respectively, and integrating the resulting equations once, it yields

$$x(t) = \frac{v_0}{\Omega_{ci}} \sin(\Omega_{ci} t) + x_0 \qquad (2.10)$$

$$y(t) = \frac{v_0}{\Omega_{ci}} \cos(\Omega_{ci} t) + y_0 \qquad (2.11)$$

Substituting $x(t = 0) = 0$ into Eq. (2.10), it yields $x_0 = 0$. Likewise, substituting $y(t = 0) = 0$ into Eq. (2.10), it yields $y_0 = -v_0/\Omega_{ci}$. Thus, we obtain the following solutions

$$x(t) = \frac{v_0}{\Omega_{ci}} \sin(\Omega_{ci} t) \qquad (2.12)$$

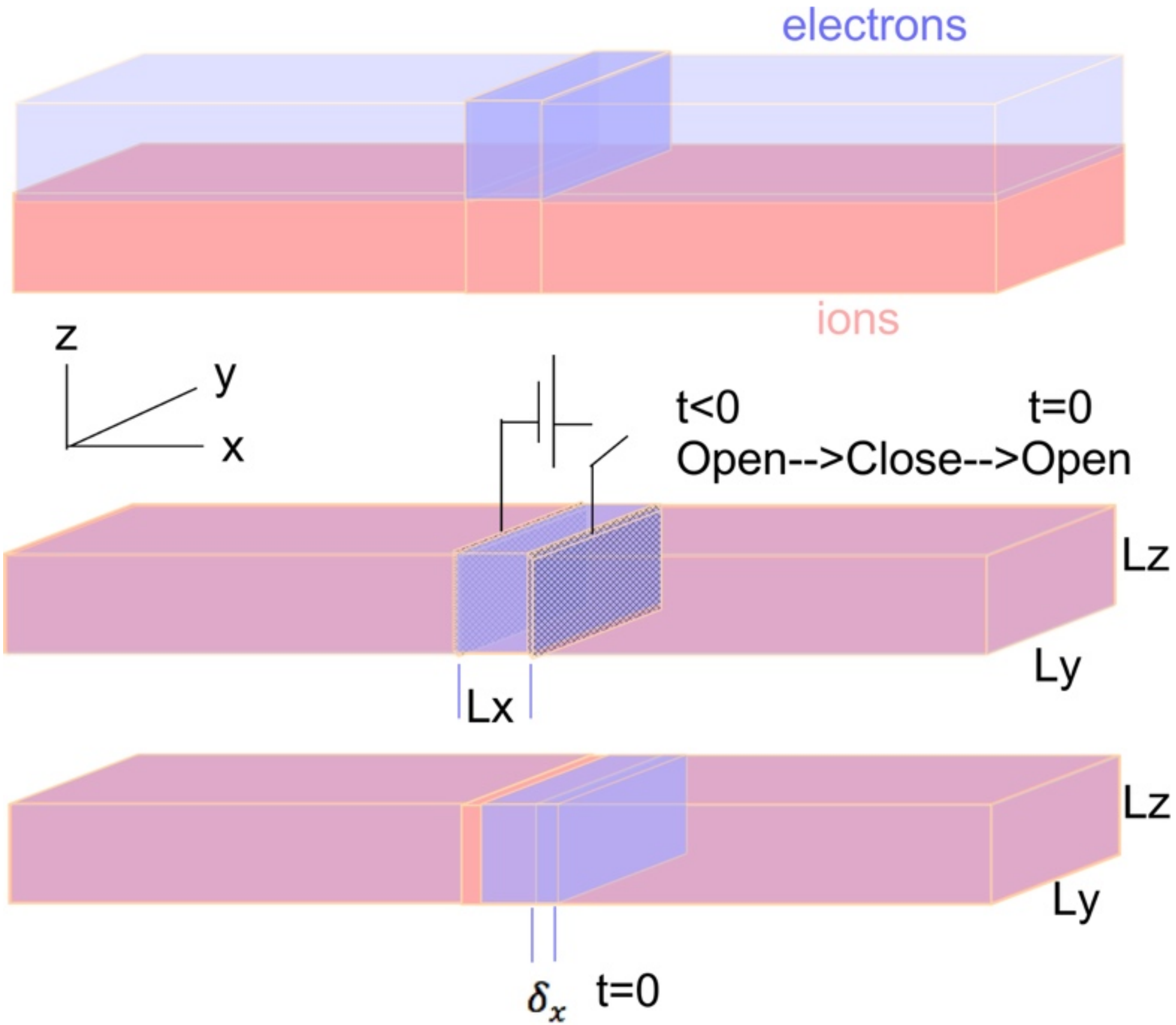
$$y(t) = \frac{v_0}{\Omega_{ci}} \cos(\Omega_{ci} t) - \frac{v_0}{\Omega_{ci}} \qquad (2.13)$$

比較第 3 步的結果與第 1 步＆第 2 步所預測的結果是否相符。如果不符，請找出哪裡出錯了。

Example 2.2.

在一團靜止不動的電漿中，插入兩片平行網狀電極，瞬間通電後，立刻斷電。假設此電漿中電子密度與正離子密度均為 n。

(a) 如果不考慮電漿中電子的熱運動，並假設正離子質量夠大，通入的電壓差不足以驅動正離子位移，則可假設正離子幾乎不動。則這團電漿中的電子，在通電、斷電後會以 $\omega_{pe} = (ne^2/\epsilon_0 m_e)^{1/2}$ 的角頻率震盪。藉由觀測 ω_{pe}，科學家可以估算電漿的 number density 個數密度 n（or 電子濃度）。


electrons
ions
z
y
x
t<0
t=0
Open-->Close-->Open
Lz
Ly
Lx
Lz
Ly
δ_x t=0

After applying an electric pulse, electrons inside the capacitor will move to the right and form a new capacitor at $t = 0$ as illustrated in the bottom panel. The Gauss law for electric field yields

$$E_x L_y L_z = \frac{en\delta_x L_y L_z}{\epsilon_0} \tag{2.14}$$

The cold electron momentum equation yields

$$m_e n L_x L_y L_z \frac{d^2\delta_x}{dt^2} = -enL_x L_y L_z E_x \tag{2.15}$$

Substituting Eq. (2.14) into (2.15) to eliminate E_x, it yields

$$\ddot{\delta}_x = -\frac{ne^2}{m_e\epsilon_0}\delta_x \tag{2.16}$$

Solution of Eq. (2.16) can be written as

$$\delta_x(t) = C_1 \cos(\omega_{pe} t) + C_2 \sin(\omega_{pe} t)$$

where $\omega_{pe} = \sqrt{ne^2/m_e\epsilon_0}$ is the electron plasma frequency (angular frequency).

Let $\delta_x(t=0) = \delta_{x0}$ and $\dot{\delta}_x(t=0) = 0$. It yields $C_1 = \delta_{x0}$ and $C_2 = 0$. Thus, the solution of Eq. (2.16) becomes

$$\delta_x(t) = \delta_{x0} \cos(\omega_{pe} t) \tag{2.17}$$

(b) 如果不考慮電漿中電子的熱運動，但考慮正離子對電場擾動產生的反應， 整個系統會以 $(\omega_{pe}^2 + \omega_{pi}^2)^{1/2}$ 的角頻率震盪

(c) 如果不考慮電漿中電子的熱運動，也不考慮正離子對電場擾動產生的反應， 但考慮電漿浸泡在強磁場 B 中，若此磁場的方向垂直於擾動電場方向，則整個系統會以 $(\omega_{pe}^2 + \Omega_{ce}^2)^{1/2}$ 的角頻率震盪，其中 $\Omega_{ce} = eB/m_e$, and $(\omega_{pe}^2 + \Omega_{ce}^2)^{1/2}$ is called upper hybrid frequency ω_{UH}. 地球電離層中的電漿，因為浸泡在磁場中，所以科學家可以藉由所觀測到的 ω_{UH} 以及已知的地球磁場強度所換算出來的 Ω_{ce} ，求得衛星所經過區域的電漿頻率，進一步估算當地電漿密度。

(d) 如果不考慮正離子運動，但考慮電子熱運動，則在壓縮時，電子的壓力變化會提供壓力梯度力，增加 restoring force，進而增加震盪角頻率。For the same amount of pressure changes, the pressure gradient force shall increase with decreasing wavelength λ. For wave number $k = 2\pi/\lambda$, the pressure gradient force shall increase with increasing k, so does the restoring force. 因此震盪角頻率會隨波數 k 增加而增大 $\omega = \sqrt{\omega_{pe}^2 + k^2 v_{the}^2}$

Example 2.3.

The average plasma density and magnetic field strength in different regions of the space environment are listed below. Find the corresponding plasma frequency and gyrofrequency of electrons and ions.

Note that

$$1\ \#/\mathrm{c.c.} = 10^6\ \#/\mathrm{m}^3$$

$$1\ \mathrm{nT} = 10^{-9}\ \mathrm{T}$$

$$m_e \approx 0.91\times10^{-30}\ \mathrm{kg}$$

$$m_i/m_e = 1836$$

$$e \approx 1.6\times10^{-19}\ \mathrm{C}$$

$$c \approx 3\times10^8\ \mathrm{m/s}$$

$$\mu_0 = 4\pi\times10^{-7}\ \mathrm{kg\cdot m\cdot C^{-2}}$$

$$\epsilon_0\mu_0 = 1/c^2$$

	locations	n_e (# /c.c.)	B (nT)	f_{pe} (Hz)	f_{pi} (Hz)	f_{ce} (Hz)	f_{ci} (Hz)
1	The base of the solar corona[1,2]	10^9	200000				
2	The solar wind at 1 AU from the Sun	5	5				
3	The shocked magnetosheath	20	20				
4	The dayside equatorial magnetosphere about 8 Re from the center of Earth	0.1	70				

5	The equatorial plasmasphere about 4 Re from the center of Earth	10^4	550				
6	The magnetotail lobes	0.01	20				
7	The near-Earth plasma sheet about 10 Re from the center of Earth	0.2	5				

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[1] https://en.wikipedia.org/wiki/Corona
[2] doi:10.1007/s11207-013-0331-7

8	The auroral acceleration region (about 0.7 Re above the ground)[3]	10	14000				
9	The low-latitude E-region ionosphere	10^6	35000				

[3] doi:10.1029/91JA02193.