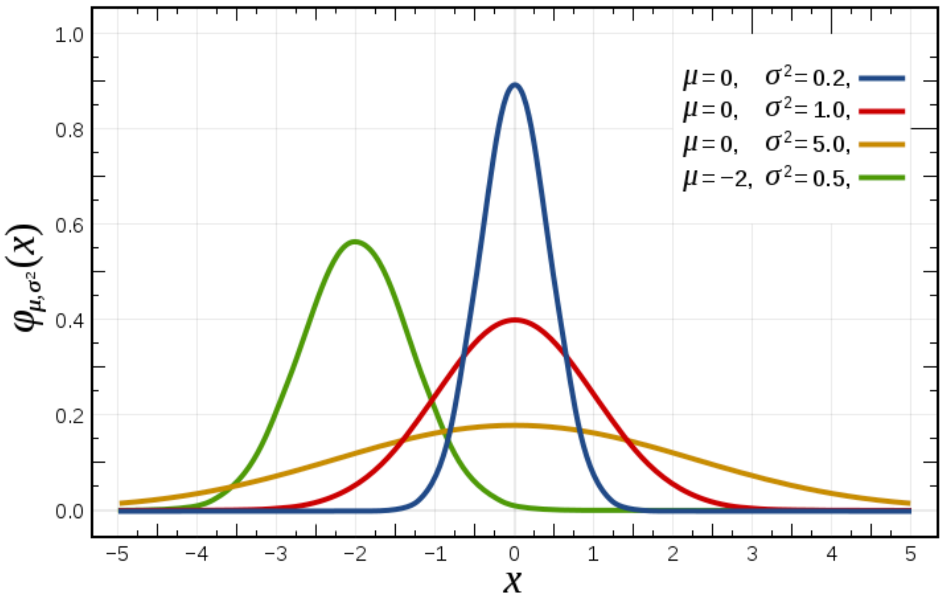
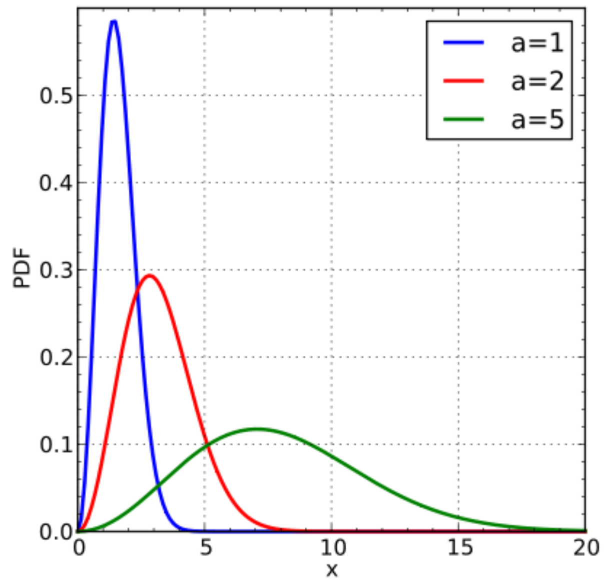
Appendix E Fluid equations
證明與應用

Normal Distribution	Maxwell-Boltzmann Distributoin
	
Page 2 https://en.wikipedia.org/wiki/Normal_distribution	$(\mu = 0,\ a = \sigma)$ Page. 20 https://en.wikipedia.org/wiki/Maxwell-Boltzmann_distribution

1-D normal distribution

$$f(x;\ \mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\,[-\frac{(x-\mu)^2}{2\sigma^2}]$$

3-D normal distribution

$$\begin{aligned}
&f_{3D}\left(x, y, z;\ \mu_x, \mu_y, \mu_z, \sigma_x, \sigma_y, \sigma_z\right) \\
&\quad = f(x; \mu_x, \sigma_x) \cdot f\left(y; \mu_y, \sigma_y\right) \cdot f(z; \mu_z, \sigma_z) \\
&\quad = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left[-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right] \cdot \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left[-\frac{\left(y-\mu_y\right)^2}{2\sigma_y^2}\right] \\
&\quad \cdot \frac{1}{\sqrt{2\pi}\sigma_z} \exp\left[-\frac{(z-\mu_z)^2}{2\sigma_z^2}\right]
\end{aligned}$$

For different (μ, σ), plot the following functions

$$f(x;\ \mu, \sigma)$$

$$xf(x;\ \mu, \sigma)$$

$$x^2 f(x;\ \mu, \sigma)$$

$$x^3 f(x;\ \mu, \sigma)$$

and determine the integration of these functions over entire x space.

Considering an αth species of a gas or a plasma has an isotropic normal distribution function in the velocity space with number density n_α, average velocity $\vec{V}_\alpha$, and thermal speed σ_α, the distribution function can be written as

$$f_\alpha\left(v_x, v_y, v_z; V_{\alpha x}, V_{\alpha y}, V_{\alpha z}, \sigma_\alpha, n_\alpha\right)$$

$$= \frac{n_\alpha}{\left(\sqrt{2\pi}\sigma_\alpha\right)^3} \exp\left[-\frac{(v_x - V_{\alpha x})^2 + \left(v_y - V_{\alpha y}\right)^2 + (v_z - V_{\alpha z})^2}{2{\sigma_\alpha}^2}\right]$$

For different $(n_\alpha, \vec{V}_\alpha, \sigma_\alpha)$, discuss the following functions' behavior

$$f_\alpha\left(v_x, v_y, v_z; V_{\alpha x}, V_{\alpha y}, V_{\alpha z}, \sigma_\alpha, n_\alpha\right)$$
$$v_x f_\alpha\left(v_x, v_y, v_z; V_{\alpha x}, V_{\alpha y}, V_{\alpha z}, \sigma_\alpha, n_\alpha\right)$$
$$v_x^2 f_\alpha\left(v_x, v_y, v_z; V_{\alpha x}, V_{\alpha y}, V_{\alpha z}, \sigma_\alpha, n_\alpha\right)$$
$$v_x^3 f_\alpha\left(v_x, v_y, v_z; V_{\alpha x}, V_{\alpha y}, V_{\alpha z}, \sigma_\alpha, n_\alpha\right)$$
$$v_x v_y f_\alpha\left(v_x, v_y, v_z; V_{\alpha x}, V_{\alpha y}, V_{\alpha z}, \sigma_\alpha, n_\alpha\right)$$
$$v_x^2 v_y f_\alpha\left(v_x, v_y, v_z; V_{\alpha x}, V_{\alpha y}, V_{\alpha z}, \sigma_\alpha, n_\alpha\right)$$

$\frac{\partial n_\alpha(\vec{x},t)}{\partial t} + \nabla \cdot [\, n_\alpha(\vec{x},t)\vec{V}_\alpha(\vec{x},t)] = 0$	(1.14)
$\frac{\partial m_\alpha n_\alpha(\vec{x},t)\vec{V}_\alpha(\vec{x},t)}{\partial t} + \nabla \cdot \left[m_\alpha n_\alpha(\vec{x},t)\vec{V}_\alpha(\vec{x},t)\,\vec{V}_\alpha(\vec{x},t) + \vec{\vec{P}}_\alpha(\vec{x},t) \right] = e_\alpha n_\alpha(\vec{x},t)[\vec{E}(\vec{x},t) + \vec{V}_\alpha(\vec{x},t)\times\vec{B}(\vec{x},t)]$	(1.15)
$\frac{\partial}{\partial t}\left[\frac{1}{2} m_\alpha n_\alpha(\vec{x},t)V_\alpha^2(\vec{x},t) + \frac{3}{2}p_\alpha(\vec{x},t)\right] + \nabla \cdot \left\{\left[\frac{1}{2} m_\alpha n_\alpha(\vec{x},t)V_\alpha^2(\vec{x},t) + \frac{3}{2}p_\alpha(\vec{x},t)\right] \vec{V}_\alpha(\vec{x},t) + \vec{\vec{P}}_\alpha(\vec{x},t) \cdot \vec{V}_\alpha(\vec{x},t) + \vec{q}_\alpha(\vec{x},t)\right\} = e_\alpha n_\alpha(\vec{x},t)\vec{E}(\vec{x},t) \cdot \vec{V}_\alpha(\vec{x},t)$	(1.16)

where $\alpha = i \; or \; e$
$n_\alpha(\vec{x},t)$ is the number density of the αth species,
$\vec{V}_\alpha(\vec{x},t)$ is the average flow velocity of the αth species,
$\overleftrightarrow{P}_\alpha(\vec{x},t)$ is the thermal pressure tensor of the αth species,
$p_\alpha(\vec{x},t)$ is the scalar pressure of the αth species,
$\vec{q}_\alpha(\vec{x},t)$ is the heat flux vector of the αth species,
and we define

$$p_\alpha(\vec{x},t) = \frac{1}{3} trace[\overleftrightarrow{P}_\alpha(\vec{x},t)] \tag{1.17}$$

也就是說，純量的熱壓 $p_\alpha(\vec{x},t)$ 定義為二階張量熱壓 $\overleftrightarrow{P}_\alpha(\vec{x},t)$ 對角線和的三分之一。

If we assume that the thermal pressure of the αth species is isotropic and assume that the change on the thermal pressure is an adiabatic process, the above fluid equations can be casted into the following simplified fluid equations: continuity equation, momentum equation, adiabatic equation of state (energy equation) of the αth species. （詳見 Lyu [2014] 第三章的推導）

$\left[\frac{\partial}{\partial t} + \vec{V}_\alpha \cdot \nabla\right] n_\alpha = -n_\alpha \nabla \cdot \vec{V}_\alpha$	(1.18)
$n_\alpha m_\alpha \left[\frac{\partial}{\partial t} + \vec{V}_\alpha \cdot \nabla\right] \vec{V}_\alpha = -\nabla p_\alpha + e_\alpha n_\alpha (\vec{E} + \vec{V}_\alpha \times \vec{B})$	(1.19)
$\left[\frac{\partial}{\partial t} + \vec{V}_\alpha \cdot \nabla\right] p_\alpha = \frac{5}{3} \frac{p_\alpha}{n_\alpha} \left[\frac{\partial}{\partial t} + \vec{V}_\alpha \cdot \nabla\right] n_\alpha$	(1.20)

$$\frac{\partial}{\partial t}\iiint f_\alpha(\vec{v},\vec{x},t)d^3v + \nabla\cdot[\iiint \vec{v} f_\alpha(\vec{v},\vec{x},t)d^3v] = 0$$	
$$\Longrightarrow \frac{\partial n_\alpha(\vec{x},t)}{\partial t} + \nabla\cdot[\, n_\alpha(\vec{x},t)\vec{V}_\alpha(\vec{x},t)] = 0$$	(1.14)
where $$n_\alpha(\vec{x},t) = \iiint f_\alpha(\vec{v},\vec{x},t)d^3v$$ $$n_\alpha(\vec{x},t)\vec{V}_\alpha(\vec{x},t) = \iiint \vec{v} f_\alpha(\vec{v},\vec{x},t)d^3v$$	

$$\frac{\partial}{\partial t}\iiint m_\alpha \vec{v} f_\alpha(\vec{v},\vec{x},t)d^3v + \nabla\cdot\left[\iiint \vec{v} m_\alpha \vec{v} f_\alpha(\vec{v},\vec{x},t)d^3v\right] = \left[\iiint e_\alpha f_\alpha(\vec{v},\vec{x},t)d^3v\right]\vec{E}(\vec{x},t) + \left[\iiint e_\alpha \vec{v} f_\alpha(\vec{v},\vec{x},t)d^3v\right]\times\vec{B}(\vec{x},t)$$	
$\Longrightarrow$ $$\frac{\partial m_\alpha n_\alpha(\vec{x},t)\vec{V}_\alpha(\vec{x},t)}{\partial t} + \nabla\cdot\left[m_\alpha n_\alpha(\vec{x},t)\vec{V}_\alpha(\vec{x},t)\,\vec{V}_\alpha(\vec{x},t) + \vec{\vec{P}}_\alpha(\vec{x},t)\right] = e_\alpha n_\alpha(\vec{x},t)[\vec{E}(\vec{x},t) + \vec{V}_\alpha(\vec{x},t)\times\vec{B}(\vec{x},t)]$$	(1.15)

where

$$\left[\iiint \vec{v} m_\alpha \vec{v} f_\alpha(\vec{v},\vec{x},t) d^3v\right]$$

$$= \iiint m_\alpha [(\vec{v}-\vec{V}_\alpha)+\vec{V}_\alpha][(\vec{v}-\vec{V}_\alpha)+\vec{V}_\alpha] f_\alpha(\vec{v},\vec{x},t) d^3v$$

$$= \left[m_\alpha n_\alpha(\vec{x},t)\vec{V}_\alpha(\vec{x},t)\,\vec{V}_\alpha(\vec{x},t) + \vec{\vec{P}}_\alpha(\vec{x},t)\right]$$

and

$$\vec{\vec{P}}_\alpha(\vec{x},t) = \iiint m_\alpha(\vec{v}-\vec{V}_\alpha)(\vec{v}-\vec{V}_\alpha) f_\alpha(\vec{v},\vec{x},t) d^3v$$

$$\frac{\partial}{\partial t}\left[\iiint \frac{1}{2} m_\alpha v^2\, f_\alpha(\vec{v},\vec{x},t) d^3v\right] + \nabla\cdot\left\{\iiint \vec{v}\,\frac{1}{2} m_\alpha v^2\, f_\alpha(\vec{v},\vec{x},t) d^3v\right\} = \left[\iiint e_\alpha \vec{v} f_\alpha(\vec{v},\vec{x},t) d^3v\right]\cdot\ \vec{E}(\vec{x},t)$$

$$\Longrightarrow \frac{\partial}{\partial t}\left[\frac{1}{2} m_\alpha n_\alpha(\vec{x},t) V_\alpha^2(\vec{x},t) + \frac{3}{2} p_\alpha(\vec{x},t)\right] + \nabla\cdot\left\{\left[\frac{1}{2} m_\alpha n_\alpha(\vec{x},t) V_\alpha^2(\vec{x},t) + \frac{3}{2} p_\alpha(\vec{x},t)\right] \vec{V}_\alpha(\vec{x},t) + \vec{\vec{P}}_\alpha(\vec{x},t)\cdot \vec{V}_\alpha(\vec{x},t) + \vec{q}_\alpha(\vec{x},t)\right\} = e_\alpha n_\alpha(\vec{x},t)\vec{V}_\alpha(\vec{x},t)\cdot\ \vec{E}(\vec{x},t) \qquad (1.16)$$

where

$$\left[\iiint \frac{1}{2} m_\alpha v^2\, f_\alpha(\vec{v},\vec{x},t) d^3v\right] = \iiint \frac{1}{2} m_\alpha \vec{v}\cdot\vec{v}\, f_\alpha(\vec{v},\vec{x},t) d^3v$$

$$= \iiint \frac{1}{2} m_\alpha \left[\left(\vec{v}-\vec{V}_\alpha\right)+\vec{V}_\alpha\right]\cdot\left[\left(\vec{v}-\vec{V}_\alpha\right)+\vec{V}_\alpha\right] f_\alpha(\vec{v},\vec{x},t) d^3v$$

$$= \left[\frac{1}{2} m_\alpha n_\alpha(\vec{x},t) V_\alpha^2(\vec{x},t) + \frac{3}{2} p_\alpha(\vec{x},t)\right]$$

$$p_\alpha(\vec{x},t) = \frac{1}{3}\iiint m_\alpha\left(\vec{v}-\vec{V}_\alpha\right)\cdot\left(\vec{v}-\vec{V}_\alpha\right) f_\alpha(\vec{v},\vec{x},t) d^3v$$

$$= \frac{1}{3} trace\left[\iiint m_\alpha\left(\vec{v}-\vec{V}_\alpha\right)\left(\vec{v}-\vec{V}_\alpha\right) f_\alpha(\vec{v},\vec{x},t) d^3v\right]$$

$$= \frac{1}{3} trace\left[\vec{\vec{P}}_\alpha(\vec{x},t)\right]$$

$$\left\{\iiint \vec{v}\,\frac{1}{2}m_\alpha v^2\, f_\alpha(\vec{v},\vec{x},t)d^3v\right\}$$

$$= \iiint \frac{1}{2}m_\alpha \vec{v}\vec{v}\cdot\vec{v}\, f_\alpha(\vec{v},\vec{x},t)d^3v$$

$$= \iiint \frac{1}{2}m_\alpha\left[\left(\vec{v}-\vec{V}_\alpha\right)+\vec{V}_\alpha\right]\left[\left(\vec{v}-\vec{V}_\alpha\right)+\vec{V}_\alpha\right]$$

$$\cdot\left[\left(\vec{v}-\vec{V}_\alpha\right)+\vec{V}_\alpha\right]f_\alpha(\vec{v},\vec{x},t)d^3v$$

$$= \left[\frac{1}{2}m_\alpha n_\alpha(\vec{x},t)V_\alpha^2(\vec{x},t)+\frac{3}{2}p_\alpha(\vec{x},t)\right]\vec{V}_\alpha(\vec{x},t)+\vec{\vec{P}}_\alpha(\vec{x},t)$$

$$\cdot\vec{V}_\alpha(\vec{x},t)+\vec{q}_\alpha(\vec{x},t)$$

$$\vec{q}_\alpha(\vec{x},t) = \iiint \frac{1}{2}m_\alpha\left(\vec{v}-\vec{V}_\alpha\right)\left(\vec{v}-\vec{V}_\alpha\right)$$

$$\cdot\left(\vec{v}-\vec{V}_\alpha\right)f_\alpha(\vec{v},\vec{x},t)d^3v$$

$(1.14) \Longrightarrow$

$$\left[\frac{\partial}{\partial t} + \vec{V}_\alpha \cdot \nabla\right] n_\alpha = -n_\alpha \nabla \cdot \vec{V}_\alpha \quad (1.18)$$

$(1.15) - m_\alpha \vec{V}_\alpha(\vec{x}, t)(1.14) \Longrightarrow$

$$\frac{\partial m_\alpha n_\alpha(\vec{x}, t)\vec{V}_\alpha(\vec{x}, t)}{\partial t} - m_\alpha \vec{V}_\alpha(\vec{x}, t)\frac{\partial n_\alpha(\vec{x}, t)}{\partial t}$$

$$+\nabla \cdot \left[m_\alpha n_\alpha(\vec{x}, t)\vec{V}_\alpha(\vec{x}, t)\,\vec{V}_\alpha(\vec{x}, t) + \vec{\vec{P}}_\alpha(\vec{x}, t)\right]$$

$$- m_\alpha \vec{V}_\alpha(\vec{x}, t)\nabla \cdot [\, n_\alpha(\vec{x}, t)\vec{V}_\alpha(\vec{x}, t)]$$

$$= e_\alpha n_\alpha(\vec{x}, t)[\vec{E}(\vec{x}, t) + \vec{V}_\alpha(\vec{x}, t)\times\vec{B}(\vec{x}, t)]$$

For isotropic pressure $\vec{\vec{P}}_\alpha(\vec{x}, t) = \vec{\vec{1}} p_\alpha(\vec{x}, t) \Longrightarrow$

$$n_\alpha m_\alpha \left[\frac{\partial}{\partial t} + \vec{V}_\alpha \cdot \nabla\right]\vec{V}_\alpha = -\nabla\, p_\alpha + e_\alpha n_\alpha(\vec{E} + \vec{V}_\alpha\times\vec{B}) \quad (1.19)$$

For isotropic pressure $\overleftrightarrow{P}_\alpha(\vec{x},t)=\overleftrightarrow{1}p_\alpha(\vec{x},t)$ and adiabatic process $\vec{q}_\alpha=0$

$$(1.16)-\frac{1}{2}m_\alpha V_\alpha^2(\vec{x},t)(1.14)-\vec{V}_\alpha(\vec{x},t)\cdot(1.19)$$

$$\Longrightarrow (1.16)-\vec{V}_\alpha(\vec{x},t)\cdot(1.15)+\frac{1}{2}m_\alpha V_\alpha^2(\vec{x},t)(1.14)$$

$$\Longrightarrow \frac{3}{2}\left[\frac{\partial}{\partial t}+\vec{V}_\alpha\cdot\nabla\right]p_\alpha+\frac{5}{2}p_\alpha\left[\nabla\cdot\vec{V}_\alpha\right]=0$$

Substituting (1.14) into the above equation to eliminate $\left[\nabla\cdot\vec{V}_\alpha\right]$, it yields

$$\left[\frac{\partial}{\partial t}+\vec{V}_\alpha\cdot\nabla\right]p_\alpha=\frac{5}{3}\frac{p_\alpha}{n_\alpha}\left[\frac{\partial}{\partial t}+\vec{V}_\alpha\cdot\nabla\right]n_\alpha \qquad (1.20)$$

Appendix E 的補充說明：

- e_α is the charge of a particle of the αth species
  例如： $e_i = e, e_e = -e$. 這裡不用 q_i, q_e 改用 e_i, e_e 是為了保留 $\vec{q}_\alpha$ 這個符號給「熱流密度向量」heat flux density.
- $trace\left[\vec{\vec{P}}_\alpha\right] = (P_{\alpha\,xx} + P_{\alpha\,yy} + P_{\alpha\,zz})$ 也就是 $\vec{\vec{P}}_\alpha$ 的矩陣表示式中，矩陣的對角線和。
- "isotropic" 就是「均向性」的意思。如果一個速度分佈函數 $f(\vec{v}) = f\left(v_x, v_y, v_z\right)$ 的平均速度為零（也就是將觀測系統轉移到平均速度的 moving frame 上做觀測）則一個均向性的速度分佈表示 $f(\vec{v}) = f(v)$ 也就說站在平均速度的

moving frame 上做觀測，不同的立體角方向，速度分佈情形相同。如為常態分佈，這樣的 isotropic 常態分佈函數有兩種表示法

第一種方法：

$$f_\alpha(\vec{v};\sigma_\alpha,n_\alpha)=f_\alpha\left(v_x,v_y,v_z;\sigma_\alpha,n_\alpha\right)$$
$$=\frac{n_\alpha}{\left(\sqrt{2\pi}\sigma_\alpha\right)^3}\exp\left[-\frac{v_x^2+v_y^2+v_z^2}{2\sigma_\alpha{}^2}\right]$$

so that

$$\iiint f_\alpha(\vec{v})d^3v=\iiint f_\alpha(\vec{v})dv_x\,dv_y\,dv_z=n_\alpha$$

第二種方法：

$$\iiint f_\alpha(\vec{v})d^3v = \iiint f_\alpha(\vec{v})v^2 \sin\theta \; dv\, d\theta\, d\phi$$

$$= \int f_\alpha(\vec{v})4\pi v^2 dv = \int f_\alpha^*(v)dv \; = n_\alpha$$

where

$$4\pi v^2 f_\alpha(\vec{v}) = 4\pi\, v^2 \frac{n_\alpha}{\left(\sqrt{2\pi}\sigma_\alpha\right)^3} \exp\left[-\frac{v_x^2 + v_y^2 + v_z^2}{2{\sigma_\alpha}^2}\right]$$

$$= 4\pi\, v^2 \frac{n_\alpha}{\left(\sqrt{2\pi}\sigma_\alpha\right)^3} \exp\left[-\frac{v^2}{2{\sigma_\alpha}^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{n_\alpha}{\sigma_\alpha^3}\; v^2 \exp\left[-\frac{v^2}{2{\sigma_\alpha}^2}\right]$$

Thus, we have

$$f_\alpha^*(v;\sigma_\alpha,n_\alpha) = \sqrt{\frac{2}{\pi}}\frac{n_\alpha}{\sigma_\alpha^3}\, v^2 \exp\left[-\frac{v^2}{2\sigma_\alpha{}^2}\right]$$

The isotropic distribution function $f_\alpha^*(v;\sigma_\alpha,n_\alpha)$ is also called Maxwell-Boltzmann distribution. (https://en.wikipedia.org/wiki/Maxwell-Boltzmann_distribution).

此函數的物理意義就是把原來三維的均向性分佈函數中同一速度的密度分佈加起來就得到 $f_\alpha^*(v;\sigma_\alpha,n_\alpha)$ 。因為速度越大，球殼面積越大，速度越小，球殼面積越小。因此 $f_\alpha^*(v;\sigma_\alpha,n_\alpha)$ 的圖形不同於 常態分佈圖型