Appendix B　Gauss Theorem (or Gauss Divergence Theorem) 證明與應用

Show that

$$\oiint_{S(Vol.)} \vec{E} \cdot d\vec{a} = \iiint_{Vol.} \frac{\rho_c}{\epsilon_0} d^3x$$

yields

$$\nabla \cdot \vec{E} = \frac{\rho_c}{\epsilon_0}$$

and vice versa (the other way around).

Let us consider an infinitesimal volume $Vol. = \Delta x \Delta y \Delta z$ centered at (x_0, y_0, z_0). The charge density inside this volume can be considered as a uniform function, thus we have

$$\iiint_{Vol.} \frac{\rho_c}{\epsilon_0} d^3x \approx \frac{\rho_c(x_0, y_0, z_0)}{\epsilon_0} \Delta x \Delta y \Delta z \qquad \text{(B.1)}$$

and the surface around the infinitesimal volume $Vol. = \Delta x \Delta y \Delta z$ should locate at $x = x_0 \pm \Delta x/2$, $y = y_0 \pm \Delta y/2$, and $z = z_0 \pm \Delta z/2$. Thus

$$
\begin{aligned}
\oiint_{S(Vol.)} \vec{E} \cdot d\vec{a}
&= E_x\left(x_0 + \frac{\Delta x}{2}, y_0, z_0\right)\Delta y \Delta z \\
&- E_x\left(x_0 - \frac{\Delta x}{2}, y_0, z_0\right)\Delta y \Delta z \\
&+ E_y\left(x_0, y_0 + \frac{\Delta y}{2}, z_0\right)\Delta x \Delta z \\
&- E_y\left(x_0, y_0 - \frac{\Delta y}{2}, z_0\right)\Delta x \Delta z \\
&+ E_z\left(x_0, y_0, z_0 + \frac{\Delta z}{2}\right)\Delta x \Delta y \\
&- E_z\left(x_0, y_0, z_0 - \frac{\Delta z}{2}\right)\Delta x \Delta y
\end{aligned}
\qquad \text{(B.2)}
$$

Substituting Equation (B.1) and (B.2) into the following equation

$$\oiint_{S(Vol.)} \vec{E} \cdot d\vec{a} = \iiint_{Vol.} \frac{\rho_c}{\epsilon_0} d^3x$$

It tyields

$$E_x\left(x_0 + \frac{\Delta x}{2}, y_0, z_0\right)\Delta y \Delta z - E_x\left(x_0 - \frac{\Delta x}{2}, y_0, z_0\right)\Delta y \Delta z$$
$$+ E_y\left(x_0, y_0 + \frac{\Delta y}{2}, z_0\right)\Delta x \Delta z - E_y\left(x_0, y_0 - \frac{\Delta y}{2}, z_0\right)\Delta x \Delta z$$
$$+ E_z\left(x_0, y_0, z_0 + \frac{\Delta z}{2}\right)\Delta x \Delta y - E_z\left(x_0, y_0, z_0 - \frac{\Delta z}{2}\right)\Delta x \Delta y$$
$$= \frac{\rho_c(x_0, y_0, z_0)}{\epsilon_0}\Delta x \Delta y \Delta z$$

Multiplying the above equation by $1/\,\Delta x \Delta y \Delta z$, it yields

$$\frac{E_x\left(x_0+\frac{\Delta x}{2},y_0,z_0\right)-E_x\left(x_0-\frac{\Delta x}{2},y_0,z_0\right)}{\Delta x}$$

$$+\frac{E_y\left(x_0,y_0+\frac{\Delta y}{2},z_0\right)-E_y\left(x_0,y_0-\frac{\Delta y}{2},z_0\right)}{\Delta y}$$

$$+\frac{E_z\left(x_0,y_0,z_0+\frac{\Delta z}{2}\right)-E_z\left(x_0,y_0,z_0-\frac{\Delta z}{2}\right)}{\Delta z}$$

$$=\frac{\rho_c(x_0,y_0,z_0)}{\epsilon_0}$$

For an infinitesimal vol. $\Delta x\,\Delta y\,\Delta z$, it yields

$$\left[\nabla\cdot\vec{E}=\frac{\rho_c}{\epsilon_0}\right]_{(x_0,y_0,z_0)}\quad Q.E.D.$$

Likewise, for

$$\nabla \cdot \vec{E} = \frac{\rho_c}{\epsilon_0}$$

it yields

$$\oiint_{S(Vol.)} \vec{E} \cdot d\vec{a} = \iiint_{Vol.} \frac{\rho_c}{\epsilon_0} d^3x$$

in an infinitesimal vol. $\Delta x\, \Delta y\, \Delta z$.

把所有這樣的小體積都加起來，就是總體積，而相鄰的「面積分」會互相抵銷，只剩下最外圍的「封閉曲面積分」。故得證 $\nabla \cdot \vec{E} = \rho_c/\epsilon_0$, 可推得以下積分形式

$$\oiint_{S(Vol.)} \vec{E} \cdot d\vec{a} = \iiint_{Vol.} \frac{\rho_c}{\epsilon_0} d^3x$$

Likewise, $\nabla \cdot \vec{B} = 0$ yields

$$\oiint_{S(Vol.)} \vec{B} \cdot d\vec{a} = 0$$

由此結果可以證明磁場線密集處磁場比較強。（請證明）

Likewise, the continuity equation

$$\left[\frac{\partial}{\partial t} + \vec{V}_\alpha \cdot \nabla\right] n_\alpha = -n_\alpha \nabla \cdot \vec{V}_\alpha$$

yields, for incompressible fluid, 流線密集處, 流速比較大,

where incompressible fluid is defined by

$$\left[\frac{\partial}{\partial t} + \vec{V}_\alpha \cdot \nabla\right] n_\alpha = \left.\frac{dn_\alpha}{dt}\right|_{along\ the\ trajectory\ of\ a\ fluid\ element} = 0$$