Lecture 10

Linear Waves in the MHD plasma: Magnetohydrodynamic Waves

磁流體 (MHD)

Key points

- MHD waves
- MHD plasma
- MHD approximation (MHD Ohm's Law)
- Ideal MHD equations
- Characteristics of MHD wave modes
- Phase velocity
- Group velocity
- Walén relation
- Field-aligned current

10.1. One-Fluid Equations

The continuity equations of the αth species given in Lecture 1 is

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \vec{V}_\alpha) = 0 \qquad (1.14\alpha)$$

$\sum_\alpha m_\alpha (1.14\alpha)$ yields the mass continuity equation of the one-fluid plasma

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \qquad (10.1)$$

$\sum_\alpha e_\alpha (1.14\alpha)$ yields the charge continuity equation of the one-fluid plasma

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot \vec{J} = 0 \qquad (10.2)$$

The momentum equations of the αth species given in Lecture 1 is

$\frac{\partial m_\alpha n_\alpha \vec{V}_\alpha}{\partial t} + \nabla \cdot (m_\alpha n_\alpha \vec{V}_\alpha \vec{V}_\alpha + \vec{\vec{P}}_\alpha) = e_\alpha n_\alpha (\vec{E} + \vec{V}_\alpha \times \vec{B})$	(1.15α)

$\sum_\alpha (1.15\alpha)$ yields the momentum equation of the one-fluid plasma

$\frac{\partial}{\partial t}(\rho \vec{V}) + \nabla \cdot (\rho \vec{V}\vec{V} + \vec{\vec{P}}) = \rho_c \vec{E} + \vec{J} \times \vec{B}$	(10.3)

$\sum_\alpha (e_\alpha / m_\alpha)(1.15\alpha)$ and for $m_e \ll m_i$, it yields the following **Generalized Omh's law**

$\frac{c^2}{\omega_{pe}^2}\mu_0 \left\{ \frac{\partial \vec{J}}{\partial t} + \nabla \cdot [(\vec{V}\vec{J} + \vec{J}\vec{V} - \rho_c \vec{V}\vec{V} - \frac{\vec{J}\vec{J}}{ne})/(1 - \frac{\rho_c}{ne})] \right\} - (\vec{E} + \vec{V} \times \vec{B}) + \frac{1}{ne}(\rho_c \vec{E} + \vec{J} \times \vec{B} - \nabla \cdot \vec{\vec{P}}_e) = 0$	(10.4)

The energy equations of the αth species given in Lecture 1 is

$\frac{\partial}{\partial t}\left(\frac{1}{2}m_\alpha n_\alpha V_\alpha^2 + \frac{3}{2}p_\alpha\right) + \nabla\cdot\left\{\left(\frac{1}{2}m_\alpha n_\alpha V_\alpha^2 + \frac{3}{2}p_\alpha\right)\vec{V}_\alpha + \vec{\vec{P}}_\alpha\cdot\vec{V}_\alpha + \vec{q}_\alpha\right\} = e_\alpha n_\alpha \vec{E}\cdot\vec{V}_\alpha$	(1.16α)

$\sum_\alpha(1.16\alpha)$ yields the energy equation of the one-fluid plasma

$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho V^2 + \frac{3}{2}p\right) + \nabla\cdot\left\{\left(\frac{1}{2}\rho V^2 + \frac{3}{2}p\right)\vec{V} + \vec{\vec{P}}\cdot\vec{V} + \vec{q}\right\} = \vec{E}\cdot\vec{J}$	(10.5)

where

$\rho = n_i m_i + n_e m_e \approx n_i m_i$	$\rho_c = e(n_i - n_e)$
$\rho\vec{V} = n_i m_i \vec{V}_i + n_e m_e \vec{V}_e$	$\vec{J} = e(n_i\vec{V}_i - n_e\vec{V}_e)$
$\rho\vec{V}\vec{V} + \vec{\vec{P}} = n_i m_i \vec{V}_i\vec{V}_i + \vec{\vec{P}}_i + n_e m_e \vec{V}_e\vec{V}_e + \vec{\vec{P}}_e$	
$\left(\frac{1}{2}\rho V^2 + \frac{3}{2}p\right)\vec{V} + \vec{\vec{P}}\cdot\vec{V} + \vec{q} = \sum_\alpha\left[\left(\frac{1}{2}m_\alpha n_\alpha V_\alpha^2 + \frac{3}{2}p_\alpha\right)\vec{V}_\alpha + \vec{\vec{P}}_\alpha\cdot\vec{V}_\alpha + \vec{q}_\alpha\right]$	

Eq. (10.1) yields

$$\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho = -\rho \, \nabla \cdot \vec{V} \qquad (10.6)$$

For isotropic pressure ($\vec{\vec{P}} = p\vec{\vec{1}}$), Eq. (10.2)$-\vec{V}$Eq.(10.3) yields

$$\rho \left(\frac{\partial}{\partial t} + \vec{V} \cdot \nabla \right) \vec{V} = - \nabla p + \rho_c \vec{E} + \vec{J} \times \vec{B} \qquad (10.7)$$

For isotropic pressure, and adiabatic process ($\nabla \cdot \vec{q} = 0$),

Eq. (10.5)$-\vec{V}$ ·Eq.(10.7)$-(1/2)\rho V^2$ Eq.(10.1) yields

$$\left(\frac{\partial}{\partial t} + \vec{V} \cdot \nabla \right) p = -\gamma p \nabla \cdot \vec{V} \qquad (10.8)$$

where $\gamma = 5/3$.

10.2. Definition of MHD-Scale Phenomena

- 磁流體現象（MHD phenomena）：
  - 空間尺度或波長遠長於正離子特徵長度：
    - 正離子流體的慣性長度 c/ω_{pi} 與
      正離子的迴旋半徑 $v_{\perp}/\Omega_{ci}$
  - 時間尺度的倒數或頻率遠低於正離子特徵頻率：
    - 正離子流體的電漿頻率 ω_{pi} 與
      正離子的迴旋頻率Ω_{ci}
- 這樣時空尺度下電漿的平均行為，可用磁流體電漿（MHD plasma）模式來描述之。

10.3. MHD-Scale MHD-Fluid and Maxwell Equations

MHD assumption = MHD Ohm's Law = $\vec{E} + \vec{V}\times\vec{B} = 0$

磁流體電漿模式包含哪些假設呢？

1. 將磁流體電漿的「長波低頻」條件帶入廣義歐姆定律（Generalized Ohm's Law）中可得 $\vec{E} + \vec{V}\times\vec{B} = 0$
   其中 $\vec{V}$ 為磁流體電漿的速度場 velocity field.
2. 頻率低，表示可以忽略位移電流項 $\epsilon_0(\partial\vec{E}/\partial t) \approx 0$，因此安培定律可寫成原始的形式 $\nabla\times\vec{B} = \mu_0\vec{J}$.

3. 因為磁流體波的「波長」遠長於電子的德拜長度 Debye Length，因此可忽略電荷分離效應，可採用「準電中性」的假設：$n_i \approx n_e$ 。也就是電荷密度 $\rho_c \approx 0.$

- 以上 1,2,3 點，是磁流體電漿模式的重要假設。

  其中第 2、3 點，也適用於正離子時空尺度的問題。

  只有第 1 點 $\vec{E} + \vec{V} \times \vec{B} = 0$ 是磁流體電漿特有的條件，故稱為 MHD approximation，也稱為磁流體歐姆定律 MHD Ohm’s Law 。

- The MHD Ohm's Law ($\vec{E} + \vec{V} \times \vec{B} = 0$) yields **Frozen-in Flux**, i.e., the magnetic flux is frozen in the plasma. Namely, it can be shown that (Appendix C in Lyu (2014))

$$\frac{d\Phi_B}{dt} = -\oint (\vec{E} + \vec{V} \times \vec{B}) \cdot d\vec{l} = 0$$

- 如果一個磁流體電漿，具有 均向性 (isotropic) 的熱壓，並遵循 等熵 (isentropic) 的絕熱過程 (adiabatic processes), 則被稱為：理想的磁流體電漿(ideal MHD plasma).

10.4. Linear waves in MHD plasma

Governing Equations of Ideal MHD plasma	Linearized Eqs.
$\frac{\partial\rho}{\partial t} + \vec{V}\cdot\nabla\rho = -\rho\,\nabla\cdot\vec{V}$	$\frac{\partial\rho_1}{\partial t} = -\rho_0\,\nabla\cdot\vec{V}_1$
$\rho\left(\frac{\partial}{\partial t} + \vec{V}\cdot\nabla\right)\vec{V} = -\,\nabla p + \vec{J}\times\vec{B}$	$\rho_0\frac{\partial\vec{V}_1}{\partial t} = -\,\nabla p_1 + \vec{J}_1\times\vec{B}_0$
$\left(\frac{\partial}{\partial t} + \vec{V}\cdot\nabla\right)p = -\gamma p\nabla\cdot\vec{V}$	$\frac{\partial p_1}{\partial t} = -\gamma p_0\nabla\cdot\vec{V}_1$
$\nabla\cdot\vec{E} \approx 0$	$\nabla\cdot\vec{E}_1 \approx 0$
$\nabla\cdot\vec{B} = 0$	$\nabla\cdot\vec{B}_1 = 0$
$\nabla\times\vec{E} = -\partial\vec{B}/\partial t$	$\nabla\times\vec{E}_1 = -\partial\vec{B}_1/\partial t$
$\nabla\times\vec{B} = \mu_0\,\vec{J}$	$\nabla\times\vec{B}_1 = \mu_0\,\vec{J}_1$
$\vec{E} + \vec{V}\times\vec{B} = 0$	$\vec{E}_1 + \vec{V}_1\times\vec{B}_0 = 0$

Fourier Transform of the Linearized Equations

$(-i\omega)\tilde{\rho}_1 = -\rho_0\, i\vec{k} \cdot \tilde{\vec{V}}_1$	(10.9)
$\rho_0(-i\omega)\tilde{\vec{V}}_1 = -\, i\vec{k}\, \tilde{p}_1 + \tilde{\vec{J}}_1 \times \vec{B}_0$	(10.10)
$(-i\omega)\tilde{p}_1 = -\gamma p_0 i\vec{k} \cdot \tilde{\vec{V}}_1$	(10.11)
$i\vec{k} \cdot \tilde{\vec{E}}_1 \approx 0$	(10.12)
$i\vec{k} \cdot \tilde{\vec{B}}_1 = 0$	(10.13)
$i\vec{k} \times \tilde{\vec{E}}_1 = -(-i\omega)\tilde{\vec{B}}_1$	(10.14)
$i\vec{k} \times \tilde{\vec{B}}_1 = \mu_0 \tilde{\vec{J}}_1$	(10.15)
$\tilde{\vec{E}}_1 = -\tilde{\vec{V}}_1 \times \vec{B}_0$	(10.16)

設法把以上方程式改寫為只含擾動速度場 $\tilde{\vec{V}}_1$的方程式

Substituting Eq. (10.16) into Eq. (10.14) to eliminate $\tilde{\vec{E}}_1$, it yields

$$i\vec{k}\times\left(-\tilde{\vec{V}}_1\times\vec{B}_0\right) = -(-i\omega)\tilde{\vec{B}}_1$$

or

$$\frac{\vec{k}}{\omega}\times(\vec{B}_0\times\tilde{\vec{V}}_1) = \tilde{\vec{B}}_1 \qquad (10.17)$$

Substituting Eq. (10.17) into Eq. (10.15) to eliminate $\tilde{\vec{B}}_1$, it yields

$$i\vec{k}\times\left[\frac{\vec{k}}{\omega}\times(\vec{B}_0\times\tilde{\vec{V}}_1)\right] = \mu_0\tilde{\vec{J}}_1 \qquad (10.18)$$

Substituting Eq. (10.18) into Eq. (10.10) to eliminate $\tilde{\vec{J}}_1$, and substituting Eq. (10.11) into Eq. (10.19) to eliminate $\tilde{p}_1$, it yields

$$\rho_0(-i\omega)\tilde{\vec{V}}_1 = -i\vec{k}\frac{-\gamma p_0\vec{k}\cdot\tilde{\vec{V}}_1}{\omega} + \frac{1}{\mu_0}\left\{i\vec{k}\times\left[\frac{\vec{k}}{\omega}\times\left(\vec{B}_0\times\tilde{\vec{V}}_1\right)\right]\right\}\times\vec{B}_0$$

or

$$\left(\frac{\omega^2}{k^2}\right)\tilde{\vec{V}}_1 = \left(-\frac{\gamma p_0}{\rho_0}\right)\hat{k}\hat{k}\cdot\tilde{\vec{V}}_1 + \frac{B_0^2}{\rho_0\mu_0}\hat{B}_0\times\left\{\hat{k}\times\left[\hat{k}\times(\hat{B}_0\times\tilde{\vec{V}}_1)\right]\right\} \quad (10.19)$$

Let $\vec{B}_0 = \hat{z}B_0, \vec{k} = k(\hat{z}\cos\theta + \hat{x}\sin\theta)$, and

$$C_{A0}^2 = \frac{B_0^2}{\mu_0\rho_0}, \quad C_{S0}^2 = \frac{\gamma p_0}{\rho_0}$$

It yields the following dispersion relation

$$\begin{bmatrix} \frac{\omega^2}{k^2} - (C_{A0}^2 + C_{S0}^2 \sin^2\theta) & 0 & -C_{S0}^2 \cos\theta \sin\theta \\ 0 & \frac{\omega^2}{k^2} - C_{A0}^2 \cos^2\theta & 0 \\ -C_{S0}^2 \cos\theta \sin\theta & 0 & \frac{\omega^2}{k^2} - C_{S0}^2 \cos^2\theta \end{bmatrix} \begin{bmatrix} \tilde{V}_{1x} \\ \tilde{V}_{1y} \\ \tilde{V}_{1z} \end{bmatrix} = 0$$

以上頻散關係 dispersion relation 可改寫為

$$\lambda \vec{\tilde{V}}_1 - \overrightarrow{\overrightarrow{M}} \cdot \vec{\tilde{V}}_1 = 0$$

where $\lambda = \omega^2/k^2$, and the matrix $\overleftrightarrow{M}$ is given by

$$\overleftrightarrow{M} = \begin{bmatrix} C_{A0}^2 + C_{S0}^2 \sin^2\theta & 0 & C_{S0}^2 \cos\theta \sin\theta \\ 0 & C_{A0}^2 \cos^2\theta & 0 \\ C_{S0}^2 \cos\theta \sin\theta & 0 & C_{S0}^2 \cos^2\theta \end{bmatrix}$$

The symmetric matrix $\overleftrightarrow{M}$ has three eigen values and three corresponding eigen vectors. It will be shown that the three eigen modes correspond to the three propagating MHD wave modes, i.e., the fast, Alfvén, and slow modes.

Solution 1: For eigen vector $\vec{\tilde{V}}_1 = \hat{y}\,\tilde{V}_{1y}$, (i.e., $\tilde{V}_{1y} \neq 0$ & $\tilde{V}_{1x} = \tilde{V}_{1z} = 0$), the eigen value should be

$$\frac{\omega^2}{k^2} = C_{A0}^2 \cos^2\theta$$

因為取 $\vec{B}_0 = \hat{z}B_0, \vec{k} = k(\hat{z}\cos\theta + \hat{x}\sin\theta)$,
故 $\vec{\tilde{V}}_1 = \hat{y}\,\tilde{V}_{1y}$ 表示 $\vec{\tilde{V}}_1$ 垂直 $\vec{B}_0$ 也垂直 $\vec{k}$ 。所以這是一個不容易衰減的橫波。這樣的 磁流體波首先被 瑞典科學家 Hannes Alfvén 提出來解釋「來自太陽的大振幅磁場擾動」波動。因此被稱為 Alfvén-mode wave.

諾貝爾獎得主

Hannes Alfvén

- 若波的速度擾動方向垂直波的前進方向（$\vec{\tilde{V}}_1 \perp \vec{k}$），則此波動是一種橫波。
- 由連續方程可知，這種波中，沒有電漿密度的擾動（$\tilde{\rho}_1 = 0$），是一種 incompressible wave.
- 橫波因為沒有密度變化，故熱耗散的問題可忽略。
- 由於 Alfvén wave 的群速度完全沿著磁場方向，所以 magnetic flux tube 就像一個導波管（wave guide），讓波可以沿著管子傳遞，因此 Alfvén wave 的振幅衰減會更小。

Phase velocity is

$$\vec{V}_{ph} = \frac{\omega}{k}\hat{k}$$

Group velocity is

$$\vec{V}_g = \frac{d\omega}{d\vec{k}} = \hat{k}\frac{\partial\omega}{\partial k} + \hat{\theta}\frac{1}{k}\frac{\partial\omega}{\partial\theta}$$

Alfvén wave

$$\vec{V}_{ph} = (\omega/k)\hat{k} = C_{A0}\cos\theta\,\hat{k}$$

$$\vec{V}_g = \frac{d\omega}{d\vec{k}} = \hat{z}\,C_{A0}$$

Alfvén wave 的 $\tilde{\vec{B}}_1$ 與 $\tilde{\vec{V}}_1$ 之間的關係：Eq. (10.17) yields

$\vec{B}_0 \dfrac{\vec{k}}{\omega} \cdot \tilde{\vec{V}}_1 - \vec{B}_0 \cdot \dfrac{\vec{k}}{\omega} \tilde{\vec{V}}_1 = \tilde{\vec{B}}_1$	(10.20)

因為 Alfvén wave $\tilde{\vec{V}}_1 \perp \vec{k}$, 因此上式中 $\vec{k} \cdot \tilde{\vec{V}}_1 = 0$。又因為

Alfvén wave $\omega / k = C_{A0} |\cos\theta|$ 故上式可改寫為

$-(\hat{k} \cdot \hat{B}_0) \dfrac{\tilde{\vec{V}}_1}{C_{A0}\cos\theta} = -\left(\dfrac{\hat{k} \cdot \hat{B}_0}{\lvert\hat{k} \cdot \hat{B}_0\rvert}\right) \dfrac{\tilde{\vec{V}}_1}{C_{A0}} = \dfrac{\tilde{\vec{B}}_1}{B_0}$	(10.21)

The relation shown in Eq.(10.21) is called Walén relation

For $\hat{k} \cdot \hat{B}_0 > 0$, $\vec{V}_1$ and $\vec{B}_1$ are out-off phase.

For $\hat{k} \cdot \hat{B}_0 < 0$, $\vec{V}_1$ and $\vec{B}_1$ are in phase。

因為 Alfvén wave 中磁場擾動與速度擾動平行或反平行（如 Walén relation），因此 Alfvén wave 中磁場擾動也會垂直背景磁場方向。所以就像圓周運動，如果加速度（速度改變量）一直垂直於原本的速度方向，則速度大小不會改變，只有方向改變。同理 Alfvén wave 中磁場大小不太改變，但是磁場方向會呈現巨大改變。

Figure 10.1 is an example of Alfvén waves observed in the solar wind reported by Belcher and Davis (1971). The perturbation magnetic field and perturbation plasma flow velocity are in phase. The perturbation on proton density and the strength of magnetic field is relatively small in comparing with the perturbations on the magnetic field components.

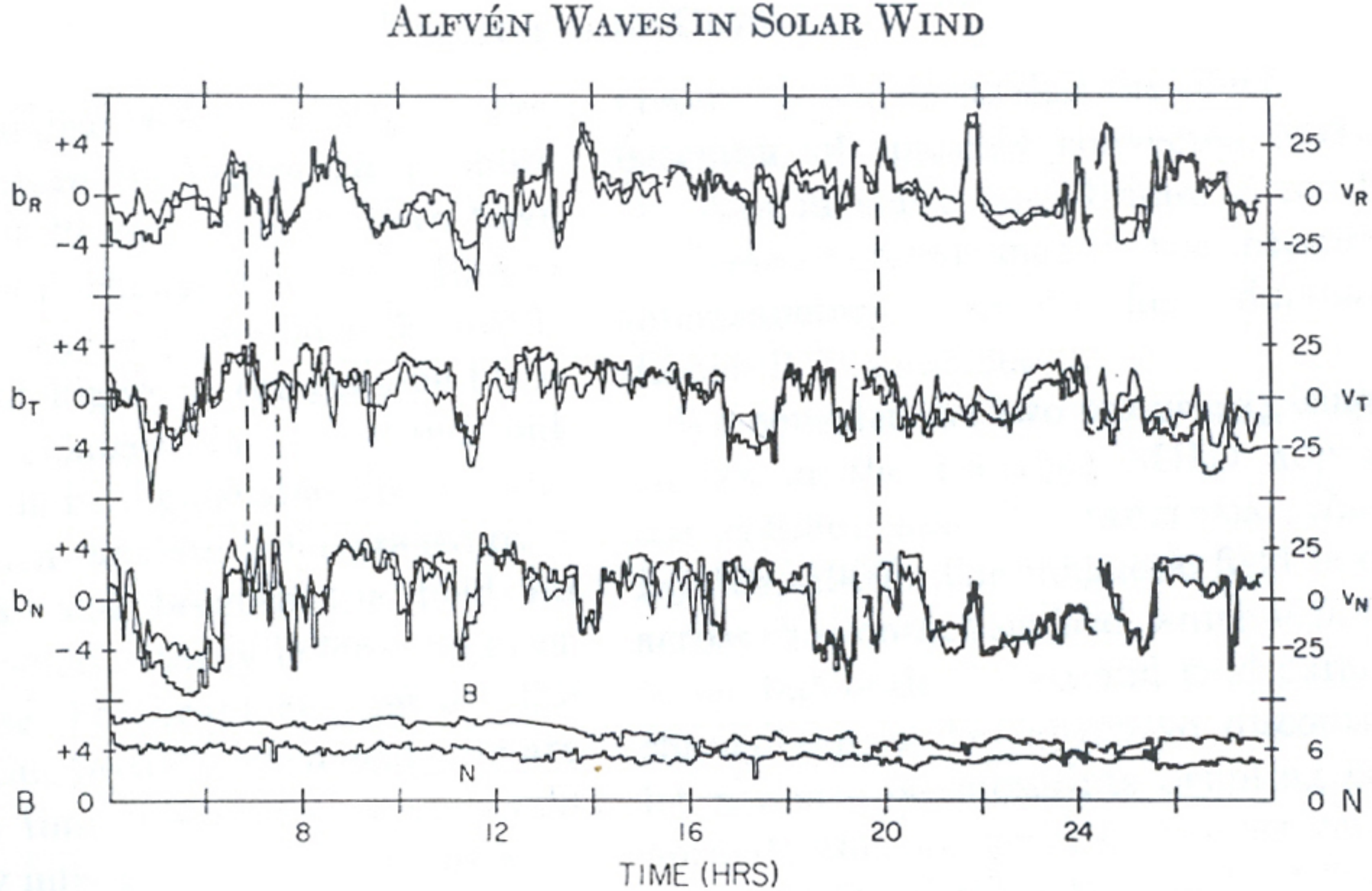

Figure 10.1. Alfvén waves observed in the solar wind by Belcher and Davis [1971].

Figure 10.2 is another example of Alfvén waves observed in the solar wind at 5AU reported by Mavromichalaki et al. (1988). Large amplitude Alfvén waves can be found in the trailing edge of the high-speed solar wind streams. The hodograms of the wave magnetic field indicate that the perturbation on the field strength is relatively small in comparing with the perturbations on the field components

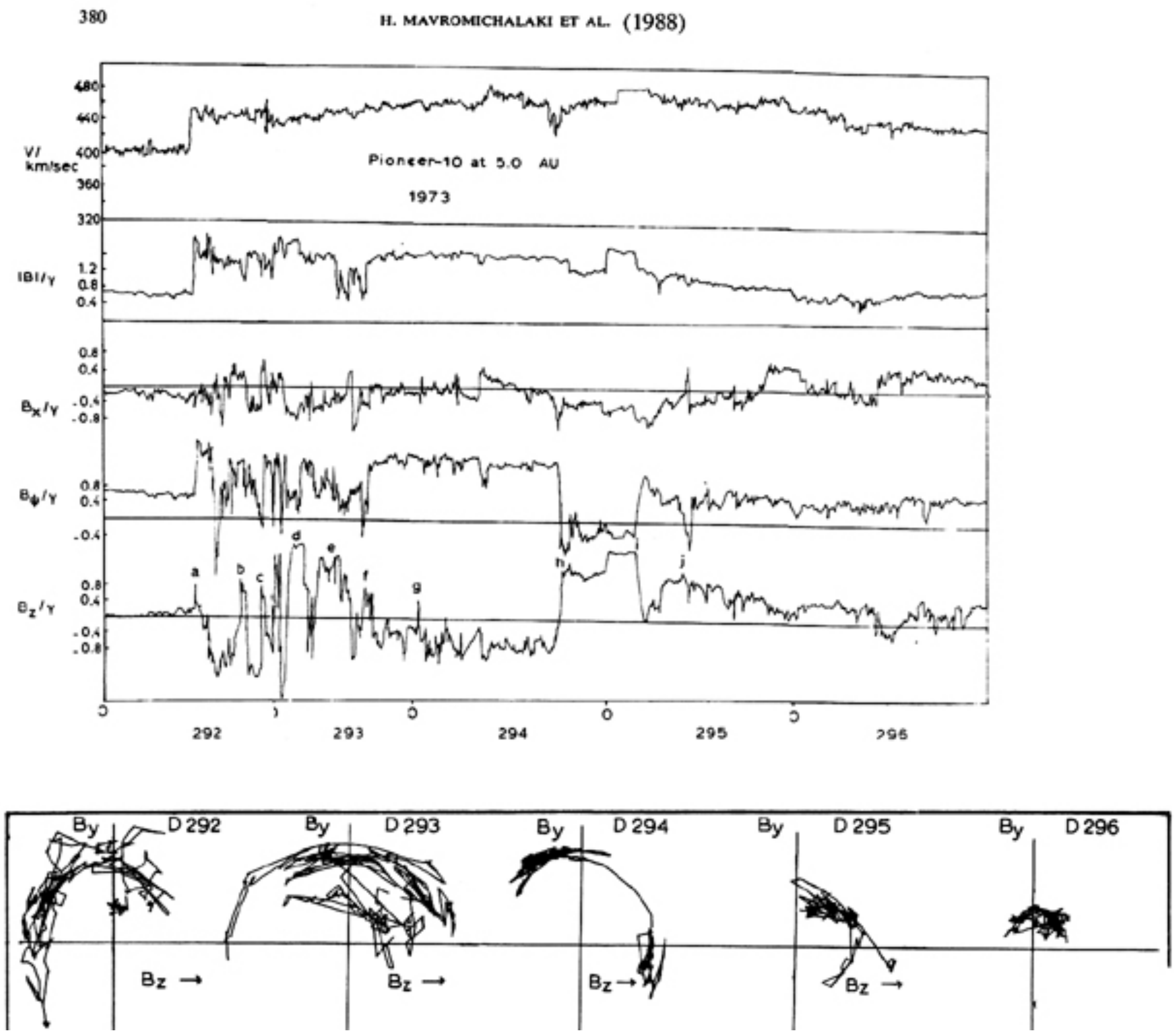

Figure 10.2. Alfven waves observed in the solar wind by Mavromichalaki et al. [1988].

Solutions 2&3: For $\tilde{\vec{V}}_1 = \hat{x}\,\tilde{V}_{1x} + \hat{z}\,\tilde{V}_{1z}$,

(i.e., $\tilde{V}_{1y} = 0$ & $(\tilde{V}_{1x}, \tilde{V}_{1z}) \neq (0,0)$), it yields

$$\det\begin{bmatrix} \dfrac{\omega^2}{k^2} - (C_{A0}^2 + C_{S0}^2 \sin^2\theta) & -C_{S0}^2 \cos\theta \sin\theta \\ -C_{S0}^2 \cos\theta \sin\theta & \dfrac{\omega^2}{k^2} - C_{S0}^2 \cos^2\theta \end{bmatrix} = 0$$

It yields

$$\left(\frac{\omega^2}{k^2}\right)_{\substack{Fast \\ Slow}} = \frac{1}{2}\left[(C_{A0}^2 + C_{S0}^2) \pm \sqrt{(C_{A0}^2 + C_{S0}^2)^2 - 4\,C_{A0}^2 C_{S0}^2 \cos^2\theta}\right]$$

其中 “+ “ 為快波的ω^2/k^2 ， “−“ 為慢波的ω^2/k^2 。

Exercise 10.1.

Plot the Friedrichs' diagram of the phase velocities of the three MHD wave modes of the following three Cases: (a) $C_{A0} = 2C_{S0}$. (b) $C_{A0} = C_{S0}$. (c) $2C_{A0} = C_{S0}$. Friedrichs' diagram of the phase velocities is a plot in the polar coordinate with $(r, \theta) = (v_{ph}, \theta_{kB})$, where θ_{kB} is the angle between wave normal $\hat{k}$ and the ambient magnetic field $\vec{B}_0$. Since $\vec{v}_{ph} = \hat{k}\omega/k$, θ_{kB} is also the angle between phase velocity $\vec{v}_{ph}$ and the ambient magnetic field $\vec{B}_0$. (Two examples are shown in Figure 10.3)

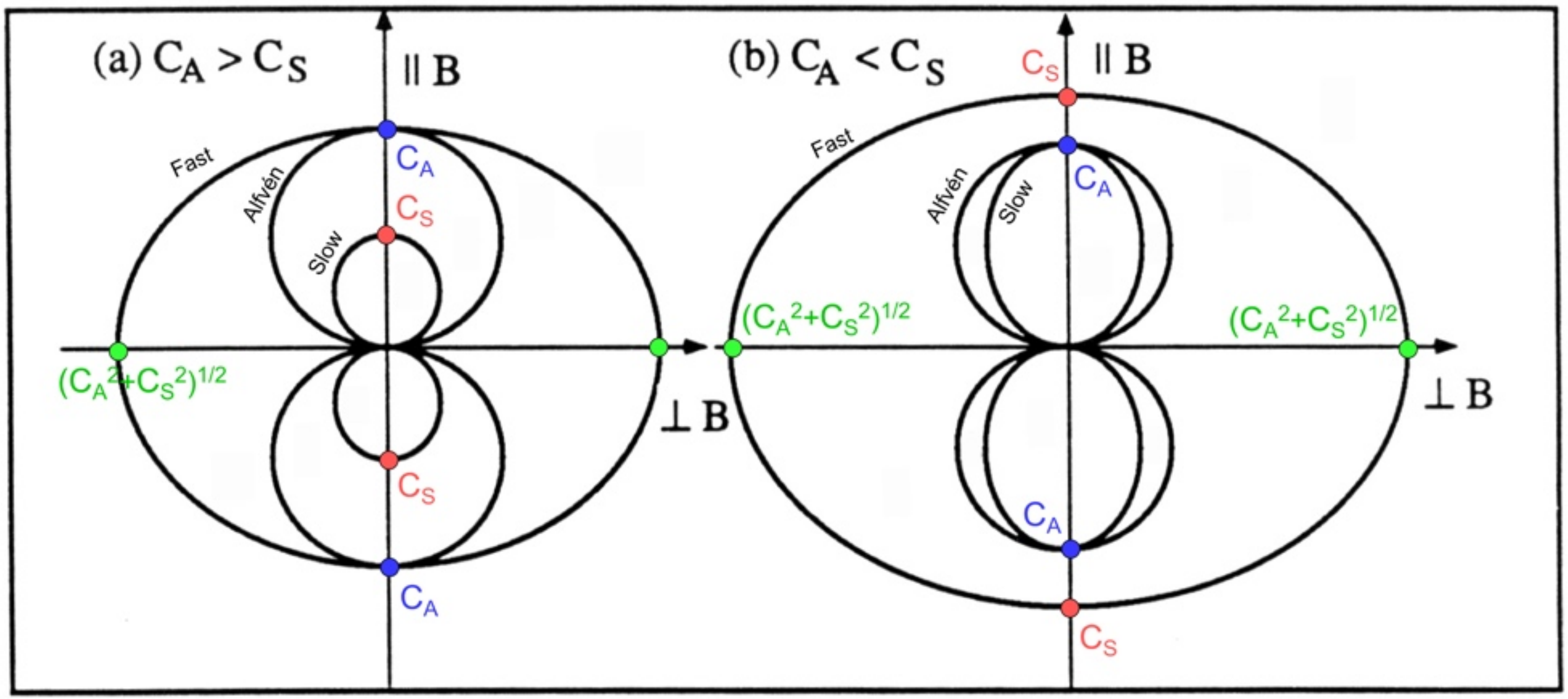

Figure 10.3. Friedrichs' diagram of the phase velocities of MHD waves

The phase velocity distributions shown in Figure 10.3 indicate that

$$\left(\frac{\omega^2}{k^2}\right)_{Fast} \geq \left(\frac{\omega^2}{k^2}\right)_{Alfvén} \geq \left(\frac{\omega^2}{k^2}\right)_{Slow}$$

Thus, Alfvén-mode wave is also called intermediate-mode wave.

補充說明：當 $C_{A0} > C_{S0}$ 時，沿磁場傳播的快波(fast-mode wave)與中速波(intermediate-mode wave) 的相速度大小相同(都是C_{A0}^2)，因此早期科學家曾經稱沿磁場傳播的「快波」為 compression Alfvén wave，而稱沿磁場傳播的「中速波」為 shear Alfvén wave。這樣的名稱，偶而還是會在文獻中出現，因此特別說明，以免同學們分不清 Alfvén wave (Alfvén-mode wave), shear Alfvén wave, compression Alfvén wave 之間的差異。

Variations of Plasma Density and Magnetic Field Strength

The phase velocity distributions shown in Figure 10.3 indicate that

$$\left(\frac{\omega^2}{k^2}\right)_{Fast} \geq C_{S0}^2 \geq \left(\frac{\omega^2}{k^2}\right)_{Slow}$$

It can be shown that if $v_{ph} > C_{S0}$, the variations of ρ_1 and B_1 will be in phase. Otherwise, if $v_{ph} < C_{S0}$, the variations of ρ_1 and B_1 will be out-off phase. (Proof of these two statements are given below.) Thus, for fast-mode wave, the variations of ρ_1 and B_1 are in phase. Whereas, for slow-mode wave, the variations of ρ_1 and B_1 are out-off phase.

觀測上，科學家常用密度與磁場強度變化為 in phase or out-off phase 來判斷 MHD wave 是快波還是慢波。

Show that, if $v_{ph} > C_{S0}$, the variations of ρ_1 and B_1 are in phase. If $v_{ph} < C_{S0}$, the variations of ρ_1 and B_1 are out-off phase.

Proof: Eqs. (10.9)~(10.11) are listed below.

$(-i\omega)\tilde{\rho}_1 = -\rho_0\, i\vec{k}\cdot\tilde{\vec{V}}_1$	(10.9)
$\rho_0(-i\omega)\tilde{\vec{V}}_1 = -\,i\vec{k}\,\tilde{p}_1 + \tilde{\vec{J}}_1\times\vec{B}_0$	(10.10)
$(-i\omega)\tilde{p}_1 = -\gamma p_0 i\vec{k}\cdot\tilde{\vec{V}}_1$	(10.11)

Eqs. (10.9) and (10.11) yields

$\tilde{p}_1 = \dfrac{\gamma p_0}{\rho_0}\tilde{\rho}_1 = C_{S0}^2\tilde{\rho}_1$	(10.19)

$i\vec{k}$ ·Eq.(10.10) yields

$\rho_0(-i\omega)i\vec{k}\cdot\tilde{\vec{V}}_1 = k^2\,\tilde{p}_1 + i\vec{k}\cdot(\tilde{\vec{J}}_1\times\vec{B}_0)$	(10.20)

Substituting Eq. (10.9) into Eq. (10.20) to eliminate $\vec{k}\cdot\tilde{\vec{V}}_1$, then substituting Eq. (10.19) into Eq. (10.20) to eliminate $\tilde{p}_1$, it yields

$\omega^2\tilde{\rho}_1 = k^2 C_{S0}^2\tilde{\rho}_1 + \vec{B}_0\cdot(i\vec{k}\times\tilde{\vec{J}}_1)$	(10.21)

Substituting Eq.(10.15) into Eq. (10.21) to eliminate $\tilde{\vec{J}}_1$ it yields

$\omega^2\tilde{\rho}_1 = k^2 C_{S0}^2\tilde{\rho}_1 + \vec{B}_0\cdot\dfrac{i\vec{k}\times\left(i\vec{k}\times\tilde{\vec{B}}_1\right)}{\mu_0}$	(10.22)

Since $i\vec{k}\times\left(i\vec{k}\times\tilde{\vec{B}}_1\right) = k^2\,\tilde{\vec{B}}_1 - \vec{k}\vec{k}\cdot\tilde{\vec{B}}_1 = k^2\,\tilde{\vec{B}}_1$,

where Eq.(10.13) has been used to set $\vec{k}\cdot\tilde{\vec{B}}_1 = 0$, it yields

$$\vec{B}_0\cdot\frac{i\vec{k}\times\left(i\vec{k}\times\tilde{\vec{B}}_1\right)}{\mu_0} = k^2\frac{\vec{B}_0\cdot\tilde{\vec{B}}_1}{\mu_0}$$

Thus, Eq. (10.22) can be rewritten in the following form

$$\left(\frac{\omega^2}{k^2} - C_{S0}^2\right)\tilde{\rho}_1 = \frac{\vec{B}_0 \cdot \tilde{\vec{B}}_1}{\mu_0} \tag{10.23}$$

The change of magnetic field strength is

$$\tilde{B}_1 = \tilde{B} - B_0 = \sqrt{\left(\vec{B}_0 + \tilde{\vec{B}}_1\right) \cdot \left(\vec{B}_0 + \tilde{\vec{B}}_1\right)} - B_0$$

$$= \sqrt{B_0^2 + 2\left(\vec{B}_0 \cdot \tilde{\vec{B}}_1\right) + \tilde{B}_1^2} - B_0$$

$$= B_0\sqrt{1 + 2\left(\vec{B}_0 \cdot \tilde{\vec{B}}_1\right)/B_0^2 + (\tilde{B}_1^2/B_0^2)} - B_0$$

$$= B_0[(1 + \left(\vec{B}_0 \cdot \tilde{\vec{B}}_1\right)/B_0^2] + O(\epsilon^2) - B_0 \approx \vec{B}_0 \cdot \tilde{\vec{B}}_1/B_0$$

where the 2nd order and higher order terms have been ignored.

Thus, Eq. (10.23) can be rewritten in the following form

$$(\frac{\omega^2}{k^2} - C_{S0}^2)\tilde{\rho}_1 = \frac{\tilde{B}_1 B_0}{\mu_0} \quad (10.24)$$

For $\omega/k > C_{S0}$, Eq. (10.24) yields $\tilde{\rho}_1/\tilde{B}_1 = \text{positive real number.}$

因為在複數平面上，正實數的主幅角為 0，因此 $\tilde{\rho}_1$ 的相位與 $\tilde{B}_1$ 的相位相同。也就是 ρ_1 and B_1 are in phase.

For $\omega/k < C_{S0}$, Eq. (10.24) yields $\tilde{\rho}_1/\tilde{B}_1 = \text{negative real number,}$

因為在複數平面上，負實數的主幅角為 π，因此 $\tilde{\rho}_1$的相位與 $\tilde{B}_1$ 的相位相差π。也就是 ρ_1 and B_1 are out-off phase.

Solution 4: Entropy mode is a non-propagating wave mode in the MHD plasma. The entropy mode is characterized by

$$p_1 = 0 \text{ but } \rho_1 \neq 0$$

In contrast, we have

Alfven mode (intermediate mode)

$$\rho_1 = p_1 = 0$$

Fast mode

$$\rho_1 \text{ and } B_1 \text{ are in phase}$$

Slow mode

$$\rho_1 \text{ and } B_1 \text{ are out-off phase}$$

Phase velocity is $\vec{V}_{ph} = (\omega/k)\hat{k}$

Group velocity is

$$\vec{V}_g = \frac{d\omega}{d\vec{k}} = \hat{k}\frac{\partial \omega}{\partial k} + \hat{\theta}\frac{1}{k}\frac{\partial \omega}{\partial \theta}$$

Fast/Slow-mode waves

$$\vec{V}_{ph} = \hat{k}\left(\frac{\omega}{k}\right)_{\substack{Fast\\Slow}} = \sqrt{\frac{1}{2}\left[(C_{A0}^2 + C_{S0}^2) \pm \sqrt{(C_{A0}^2 + C_{S0}^2)^2 - 4\,C_{A0}^2 C_{S0}^2 \cos^2\theta}\right]}\,\hat{k}$$

$$\omega^2 = k^2 v_{ph}^2(\theta)$$

$$\rightarrow 2\omega\frac{\partial \omega}{\partial k} = 2k v_{ph}^2(\theta) \rightarrow \frac{\partial \omega}{\partial k} = \frac{v_{ph}^2(\theta)}{\omega/k} = v_{ph}(\theta)$$

$$\rightarrow 2\omega\frac{\partial \omega}{\partial \theta} = k^2\frac{d v_{ph}^2(\theta)}{d\theta} \rightarrow \frac{1}{k}\frac{\partial \omega}{\partial \theta} = \frac{1}{\omega/k}\frac{d v_{ph}^2(\theta)}{d\theta} = \frac{1}{v_{ph}(\theta)}\frac{d v_{ph}^2(\theta)}{d\theta}$$

Exercise 10.2.

Plot the Friedrichs' diagram of the group velocities of the three MHD wave modes of the following three Cases: (a) $C_{A0} = 2C_{S0}$. (b) $C_{A0} = C_{S0}$. (c) $2C_{A0} = C_{S0}$. Friedrichs' diagram of the phase velocities is a plot in the polar coordinate with $(r, \theta) = (v_g, \theta_{v_g B})$, where $\theta_{v_g B}$ is the angle between group velocity $\vec{v}_g$ and the ambient magnetic field $\vec{B}_0$.

Applications of the Friedrichs' diagram of the group velocities of the three MHD propagating wave modes are given below.

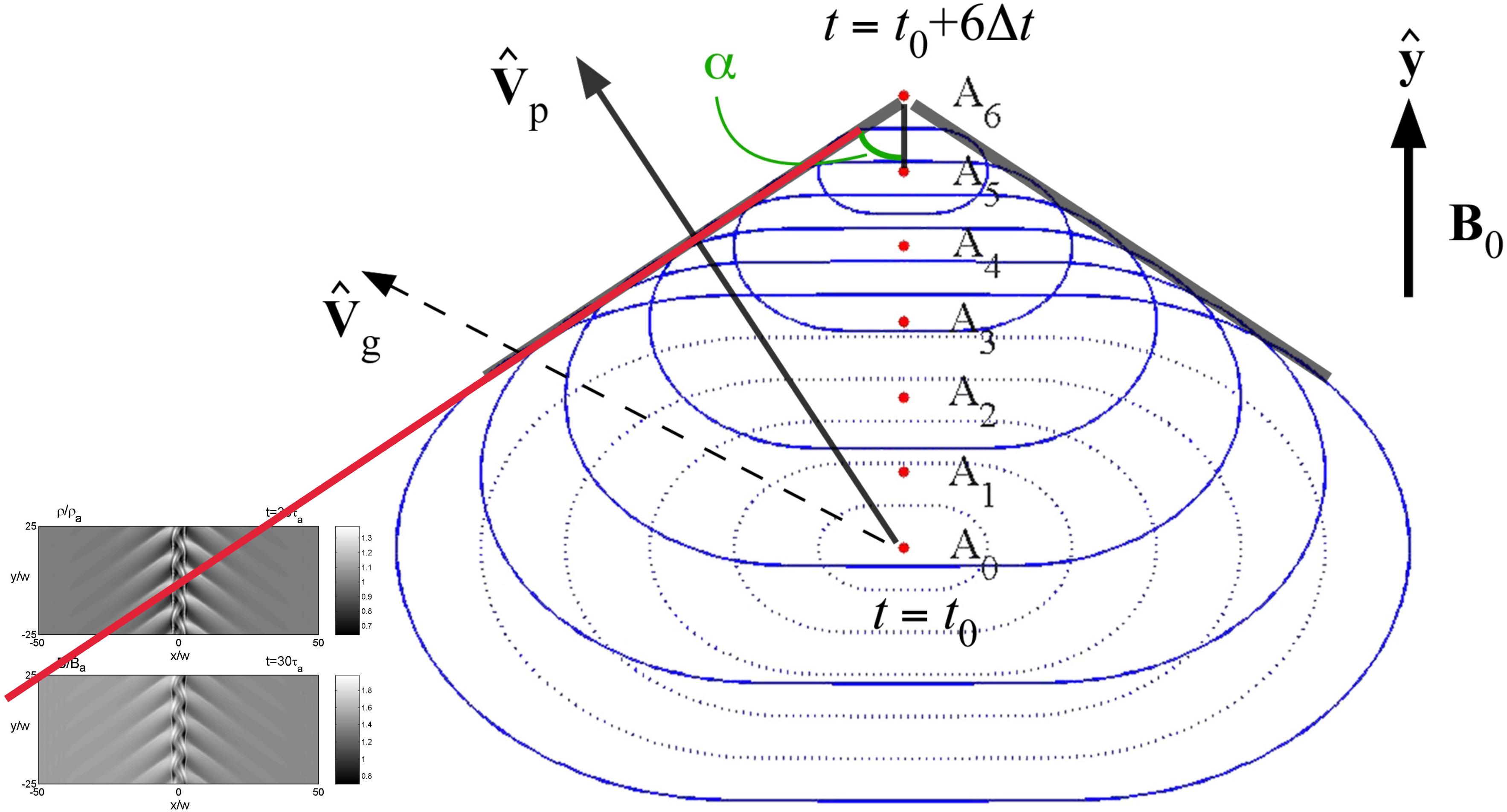

Fig. 10.4. Fast-mode Mach-cone wave (Adapted from Lai and Lyu, 2008)

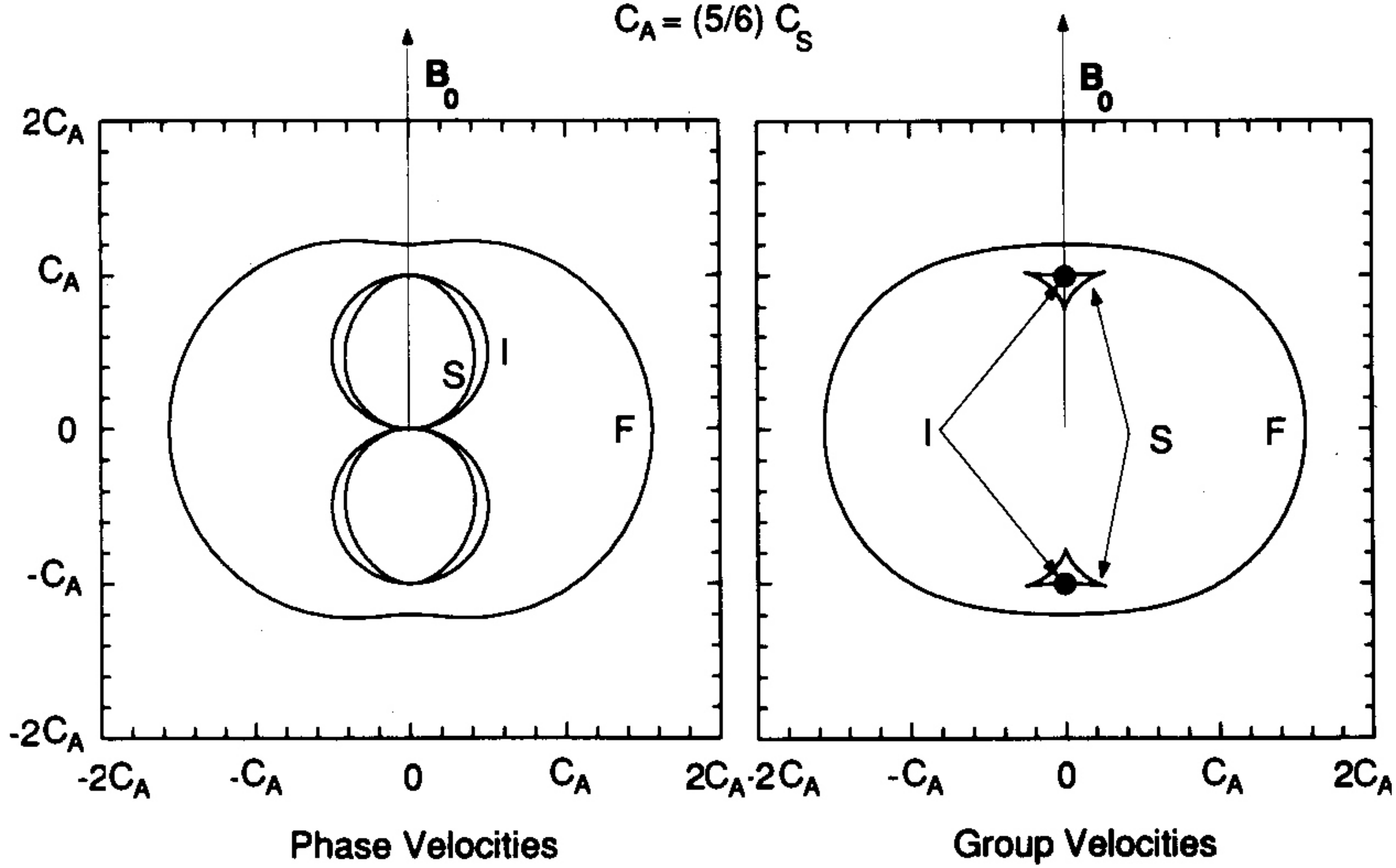

Figure 10.5. Friedrichs' diagram of the phase velocities and group velocity of MHD waves (Adapted from Kivelson, 1995)

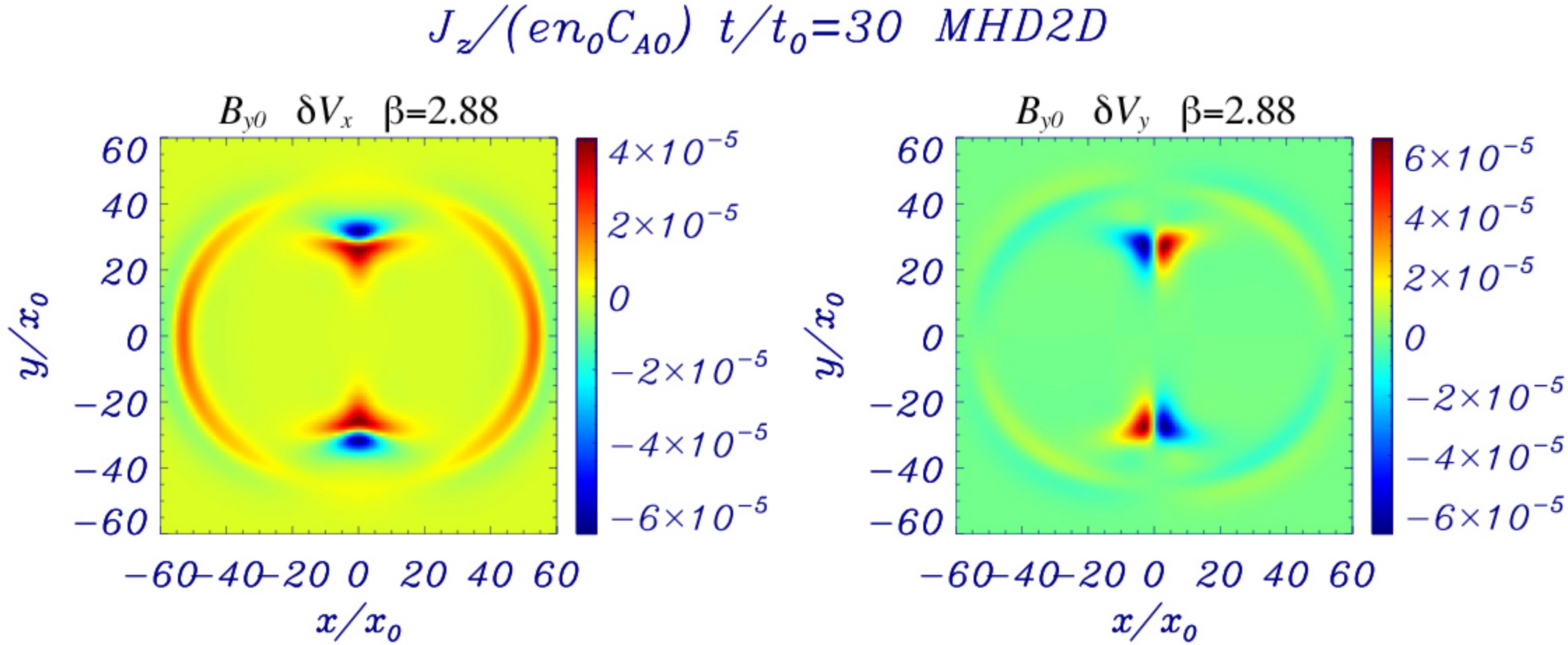

Figure 10.6. Simulation results of MHD waves with a localized disturbance at x=y=0. Comparing the results shown in Figures 10.5 and 10.6, the disturbances should expand based on the group velocity but not the phase velocity.

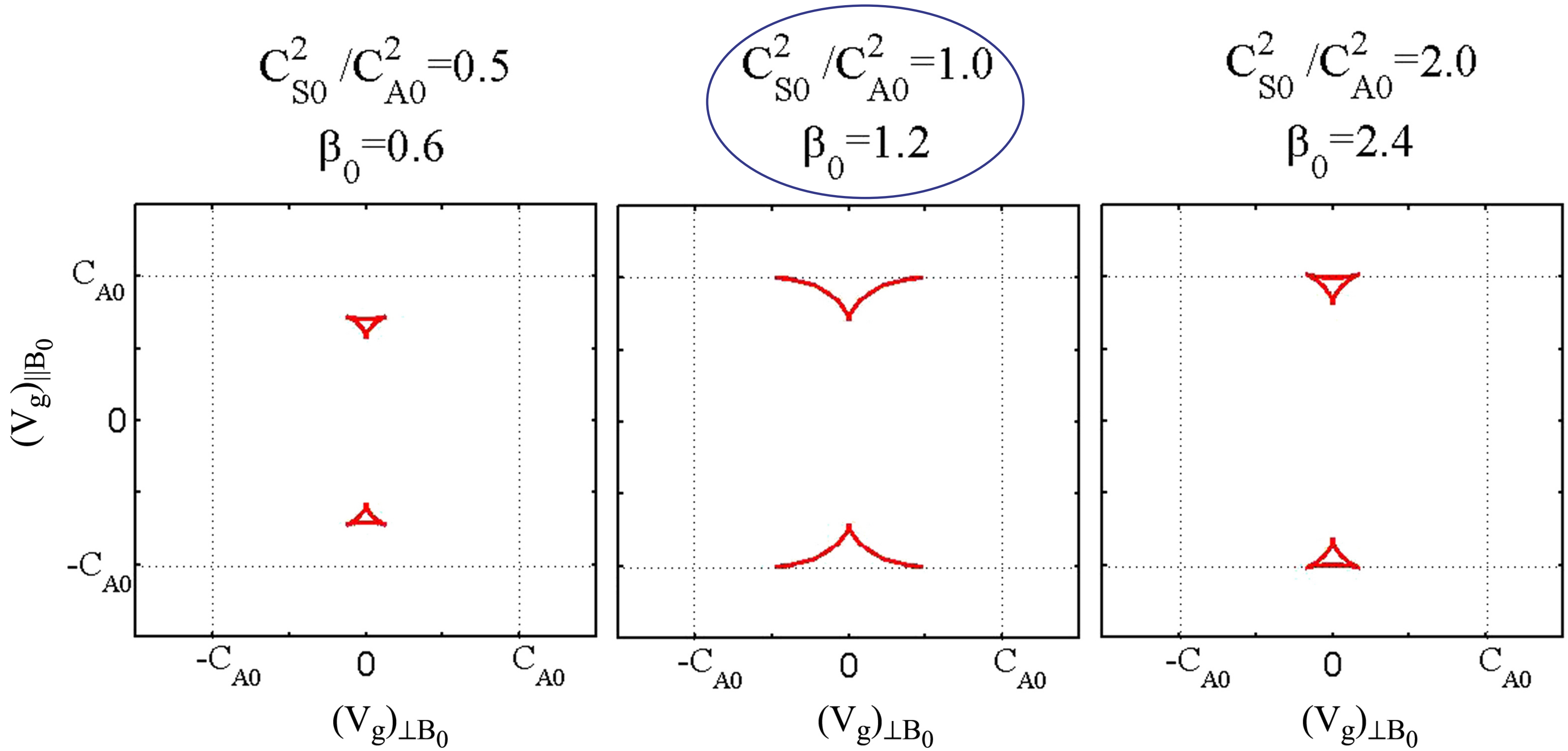

Figure 10.7. Friedrichs' diagram of the group velocity of the slow mode waves at different plasma beta. (Adapted from Lai and Lyu, 2008)

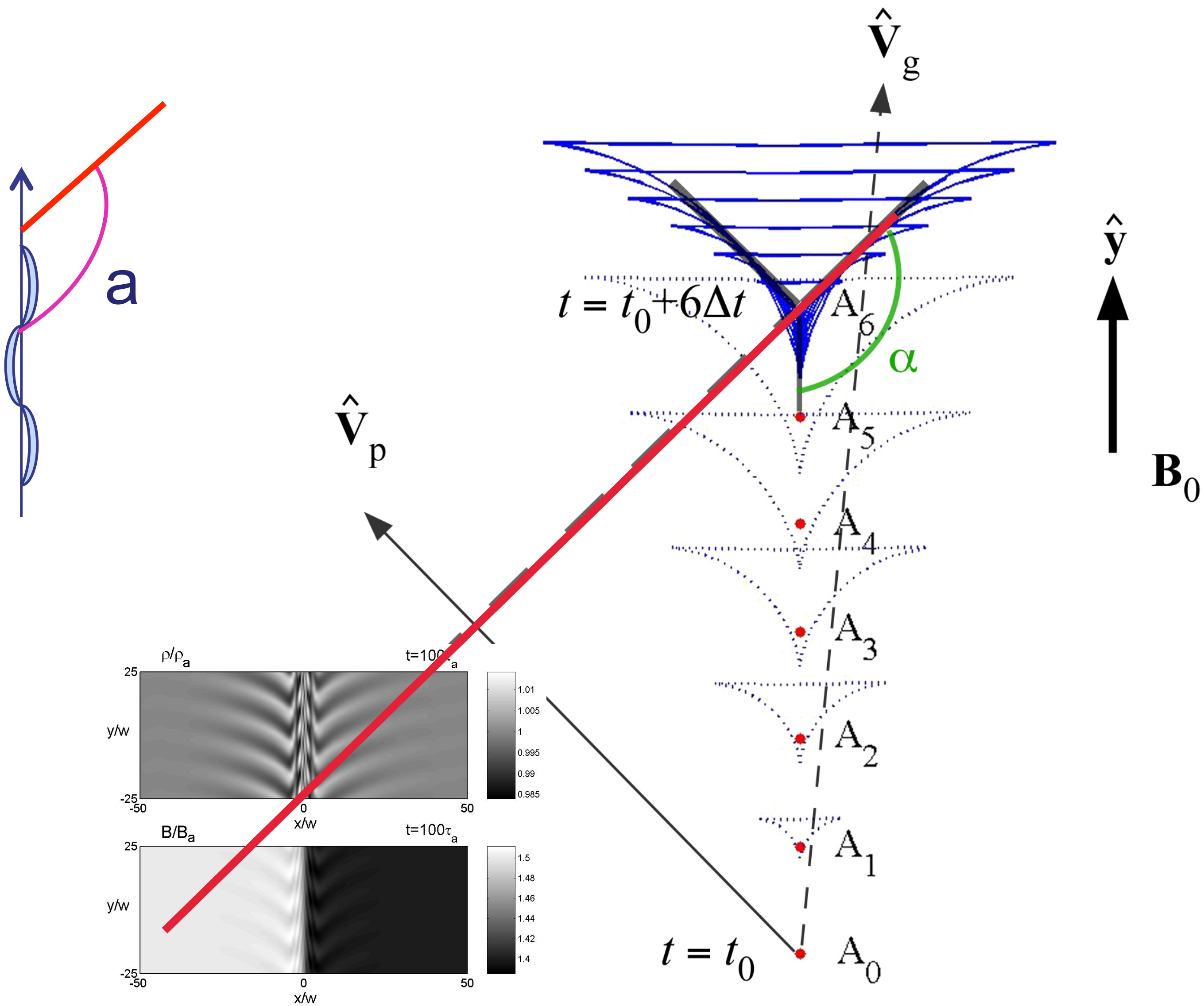

Figure 10.8. Slow-mode Mach-cone wave (Adapted from Lai and Lyu, 2008)

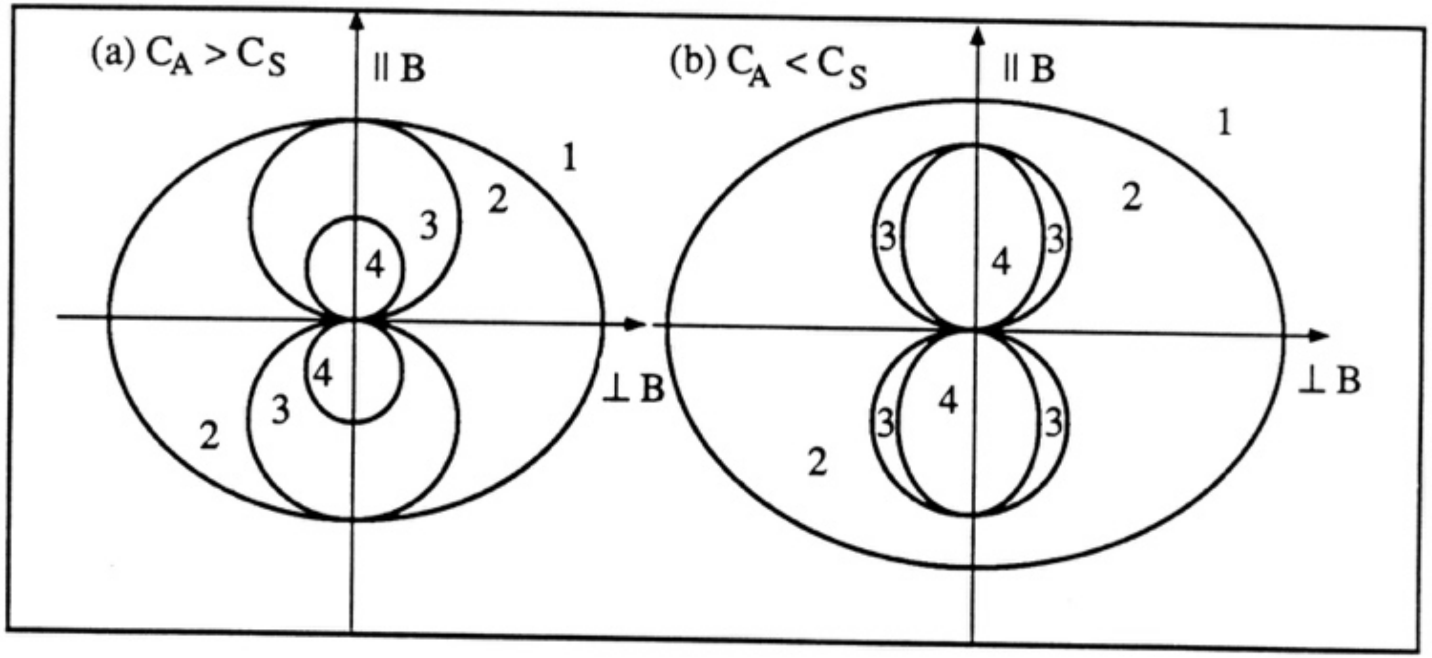

Figure 2.1 Friedrichs' diagrams of MHD waves with (*a*) $C_A > C_S$ and (*b*) $C_A < C_S$. The Friedrichs' diagram displays the dependence of three MHD wave phase speeds (radial coordinate) on their angle of propagation with respect to the ambient magnetic field (the angular coordinate measured from the vertical axis). MHD shock waves can be found with upstream state in areas 1, 2, and 3 and with downstream state in areas 2, 3, and 4 as listed in Table **2.1**.

Table 2.1 Upstream and downstream normal flow speed of MHD shock waves

Shock Types	Brief Notations	Upstream Flow Speed	Downstream Flow Speed
Fast	$1 \rightarrow 2$	$V_{F1} < V_{N1}$	$V_{AX2} < V_{N2} < V_{F2}$
Fast-Alfvén	$1 \rightarrow 3$	$V_{F1} < V_{N1}$	$V_{SL2} < V_{N2} < V_{AX2}$
Fast-Alfvén-Slow	$1 \rightarrow 4$	$V_{F1} < V_{N1}$	$V_{N2} < V_{SL2}$
Alfvén	$2 \rightarrow 3$	$V_{AX1} < V_{N1} < V_{F1}$	$V_{SL2} < V_{N2} < V_{AX2}$
Alfvén-Slow	$2 \rightarrow 4$	$V_{AX1} < V_{N1} < V_{F1}$	$V_{N2} < V_{SL2}$
Slow	$3 \rightarrow 4$	$V_{SL1} < V_{N1} < V_{AX1}$	$V_{N2} < V_{SL2}$

V_{N1} is the normal upstream flow speed in the shock rest frame.
V_{N2} is the normal downstream flow speed in the shock rest frame.
Brief notations are obtained based on upstream and downstream states of each shock and the corresponding areas shown in Figure 2.1.

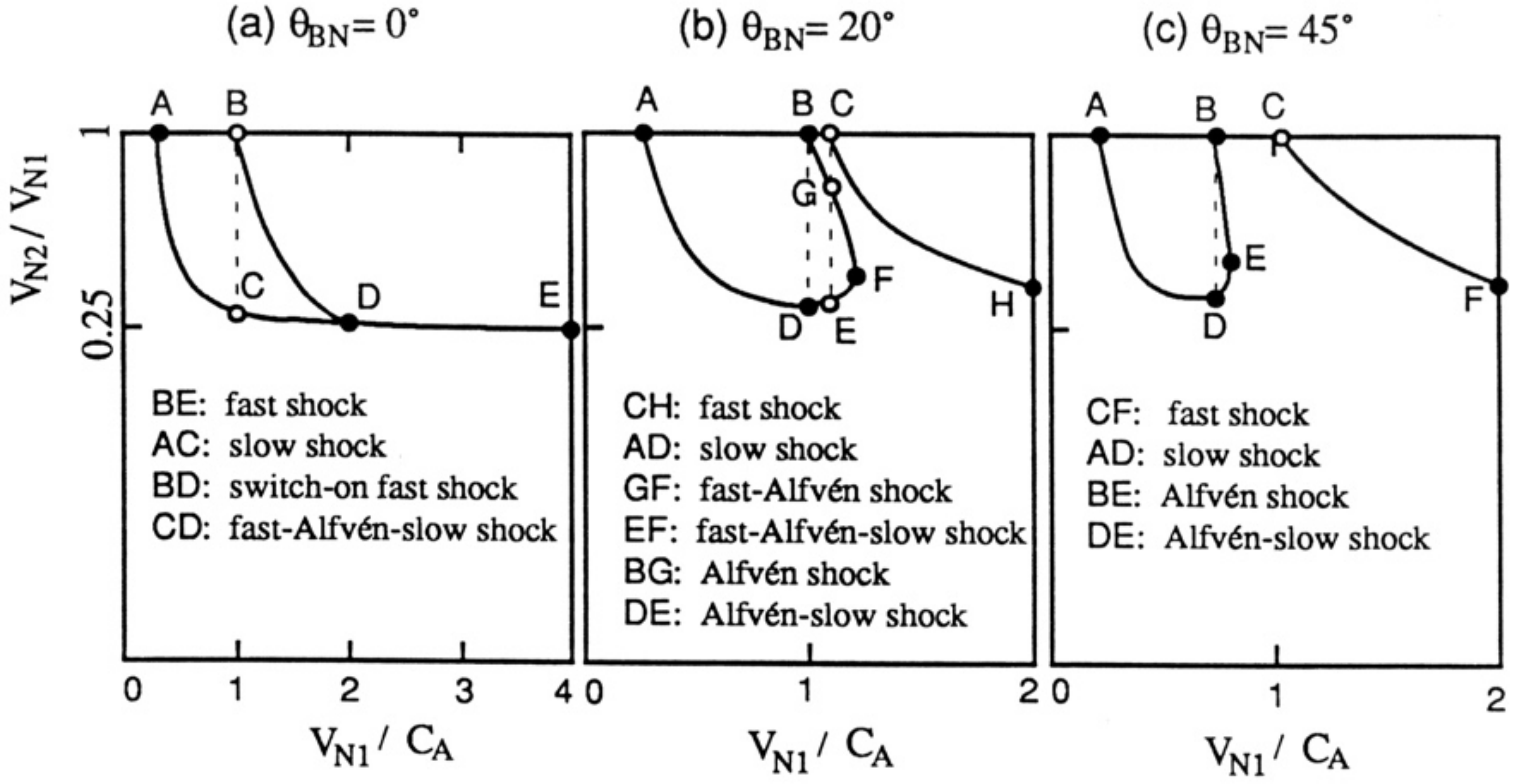

Figure 2.3 Velocity jump across various types of MHD shocks for $\beta = 0.1$, $\theta_{BN} =$ (*a*) 0°, (*b*) 20° and (*c*) 45°, where V_{N1} and V_{N2} are the upstream and downstream normal flow speed in the shock rest frame, respectively.

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