Lecture 9 Linear Waves in the Ion-Electron Two-Fluid Plasma

Key points

- Electron-time-scale linear waves in an unmagnetized plasma
  - Langmuir waves
  - Cut-off frequency, resonant frequency
- Electron time-scale linear waves in a magnetized plasma
  - Mobility tensor
  - L-mode, R-mode
  - O-mode, X-mode, Z-mode
- Ion-time-scale linear waves
  - Ion acoustic waves
  - Whistler waves, Chorus waves

9.0. Review of fluid equations of the αth species discussed in Lecture 1

$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \vec{V}_\alpha) = 0$	(1.14)
$\frac{\partial m_\alpha n_\alpha \vec{V}_\alpha}{\partial t} + \nabla \cdot \left(m_\alpha n_\alpha \vec{V}_\alpha \vec{V}_\alpha + \vec{\vec{P}}_\alpha \right) = e_\alpha n_\alpha (\vec{E} + \vec{V}_\alpha \times \vec{B})$	(1.15)
$\frac{\partial}{\partial t}\left(\frac{1}{2} m_\alpha n_\alpha V_\alpha^2 + \frac{3}{2} p_\alpha \right)$ $+\nabla \cdot \left[\left(\frac{1}{2} m_\alpha n_\alpha V_\alpha^2 + \frac{3}{2} p_\alpha \right) \vec{V}_\alpha + \vec{\vec{P}}_\alpha \cdot \vec{V}_\alpha + \vec{q}_\alpha \right] = e_\alpha n_\alpha \vec{E} \cdot \vec{V}_\alpha$	(1.16)

It can be shown that, for isotropic pressure ($\vec{\vec{P}}_\alpha = \vec{\vec{1}} p_\alpha$), Eqs. (1.14)~(1.16) can be written in the following form.

Eq. (1.14) yields

$$\left(\frac{\partial}{\partial t} + \vec{V}_\alpha \cdot \nabla\right) n_\alpha = -n_\alpha \nabla \cdot \vec{V}_\alpha \tag{1.18}$$

Eq. (1.15) $-m_\alpha \vec{V}_\alpha$ Eq. (1.14) yields

$$n_\alpha m_\alpha \left(\frac{\partial}{\partial t} + \vec{V}_\alpha \cdot \nabla\right)\vec{V}_\alpha = -\nabla p_\alpha + e_\alpha n_\alpha (\vec{E} + \vec{V}_\alpha \times \vec{B}) \tag{1.19}$$

Eq. (1.16)$-\vec{V}_\alpha$ ·Eq. (1.19)$-(1/2) m_\alpha V_\alpha^2$ Eq. (1.14) yields

$$\frac{3}{2}\left(\frac{\partial}{\partial t} + \vec{V}_\alpha \cdot \nabla\right) p_\alpha = -\frac{5}{2} p_\alpha \nabla \cdot \vec{V}_\alpha - \nabla \cdot \vec{q}_\alpha \tag{9.0.1}$$

For adiabatic process ($\nabla \cdot \vec{q}_\alpha = 0$), the above energy equation of the αth species (9.0.1) is reduced to

$$\left(\frac{\partial}{\partial t} + \vec{V}_\alpha \cdot \nabla\right) p_\alpha = -\frac{5}{3} p_\alpha \nabla \cdot \vec{V}_\alpha \tag{9.0.2}$$

Substituting Eq. (1.18) into Eq. (9.0.2) to eliminate $\nabla \cdot \vec{V}_\alpha$, it yields

$$\left(\frac{\partial}{\partial t} + \vec{V}_\alpha \cdot \nabla\right) p_\alpha = \frac{5}{3} \frac{p_\alpha}{n_\alpha} \left(\frac{\partial}{\partial t} + \vec{V}_\alpha \cdot \nabla\right) n_\alpha \quad (1.20)$$

Eq. (1.20) can be rewritten in "a constant of equation of state"

$$\left(\frac{\partial}{\partial t} + \vec{V}_\alpha \cdot \nabla\right) \ln \left(p_\alpha n_\alpha^{-\gamma_\alpha}\right) = 0 \quad (9.0.3)$$

For $\gamma_\alpha = 5/3$, it implies $\nabla \cdot \vec{q}_\alpha = 0$. Thus, it is called the adiabatic equation of state.

For isothermal process, where $\nabla \cdot \vec{q}_\alpha \neq 0$, we have $\gamma_\alpha = 1$.

Multiplying Eq. (9.0.3) by p_α, it yields

$$\left(\frac{\partial}{\partial t} + \vec{V}_\alpha \cdot \nabla\right) p_\alpha = \frac{\gamma_\alpha p_\alpha}{n_\alpha} \left(\frac{\partial}{\partial t} + \vec{V}_\alpha \cdot \nabla\right) n_\alpha = -\gamma_\alpha p_\alpha \nabla \cdot \vec{V}_\alpha \quad (9.0.4)$$

9.1. The ion-electron two-fluid equations and Maxwell equations

The continuity equations of the electrons and ions

$\left(\frac{\partial}{\partial t} + \vec{V}_e \cdot \nabla\right) n_e = -n_e \nabla \cdot \vec{V}_e$	(9.1)
$\left(\frac{\partial}{\partial t} + \vec{V}_i \cdot \nabla\right) n_i = -n_i \nabla \cdot \vec{V}_i$	(9.2)

The momentum equations of the electrons and ions

$m_e n_e \left(\frac{\partial}{\partial t} + \vec{V}_e \cdot \nabla\right) \vec{V}_e = -\nabla p_e - e n_e (\vec{E} + \vec{V}_e \times \vec{B})$	(9.3)
$m_i n_i \left(\frac{\partial}{\partial t} + \vec{V}_i \cdot \nabla\right) \vec{V}_i = -\nabla p_i + e n_i (\vec{E} + \vec{V}_i \times \vec{B})$	(9.4)

The energy equations (the equations of the state) of the electrons and ions

$\left(\frac{\partial}{\partial t}+\vec{V}_e\cdot\nabla\right)p_e=\frac{\gamma_e p_e}{n_e}\left(\frac{\partial}{\partial t}+\vec{V}_e\cdot\nabla\right)n_e=-\gamma_e p_e\nabla\cdot\vec{V}_e$	(9.5)
$\left(\frac{\partial}{\partial t}+\vec{V}_i\cdot\nabla\right)p_i=\frac{\gamma_i p_i}{n_i}\left(\frac{\partial}{\partial t}+\vec{V}_i\cdot\nabla\right)n_i=-\gamma_i p_i\nabla\cdot\vec{V}_i$	(9.6)

where $1\le\gamma_e\le 5/3$. For electron-time-scale phenomena we can assume that $\gamma_e\approx 5/3$, and ions are not moves in the electrons' time scale. For ion-time-scale phenomena we can assume that $\gamma_e\approx 1, \gamma_i\approx 5/3, n_e\approx n_i$, and ignore the electrons' inertial term, because in the ion time scale, $O\left[m_e n_e\left(\partial\vec{V}_e/\partial t\right)\right]\ll O(en_e\vec{V}_e\times B)$.

The Maxwell equations in the ion-electron two-fluid plasma

$\nabla \cdot \vec{E} = \frac{e(n_i - n_e)}{\epsilon_0}$	(9.7)
$\nabla \cdot \vec{B} = 0$	(9.8)
$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	(9.9)
$\nabla \times \vec{B} = \mu_0 e(n_i \vec{V}_i - n_e \vec{V}_e) + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$	(9.10)

Review: How to linearize the equations

Step 1: 先將所有變數拆解為平衡態 + 擾動態

$$A(\vec{x}, t) = A_0(\vec{x}) + A_1(\vec{x}, t)$$

Step 2: 寫出平衡態的流體方程式，並依問題決定平衡態特性。

Step 3: 寫出平衡態 + 擾動態的流體方程式

Step 4: 將 Step 3 的結果減去 Step 2 的結果，得擾動態方程式

Step 5: 若$O(A_1/A_0) = O(\epsilon) < 10^{-3}$，則消去等於或小於 $O(\epsilon^2)$, 的非線性項，就可得到線性化的擾動態方程式。

有關平衡態的補充說明：

從流體的觀點看，如果背景場的分佈有一個梯度，就有可能是一個不穩定的平衡態(unstable equilibrium)。反之，如果背景場在空間中是均勻分佈的，則此背景場通常是一個穩定的平衡態(stable equilibrium)。不過，如果進一步考慮 Kinetic effect 則只有當速度空間分佈，沒有高低起伏的 local minimum 分部時，才能保證系統是處於線性的穩定平衡態(stable equilibrium) 。

9.2. Uniform background equilibrium

For simplicity, we assume that the background equilibrium is uniform. Namely, $A(\vec{x},t) = A_0 + A_1(\vec{x},t)$

Let us examine the equilibrium equations

$(\vec{V}_{e0} \cdot \nabla) n_{e0} = -n_{e0} \nabla \cdot \vec{V}_{e0}$	$0 = 0$
$(\vec{V}_{i0} \cdot \nabla) n_{i0} = -n_{i0} \nabla \cdot \vec{V}_{i0}$	$0 = 0$
$m_e n_{e0} (\vec{V}_{e0} \cdot \nabla) \vec{V}_{e0}$ $= -\nabla p_{e0} - e n_{e0} (\vec{E}_0 + \vec{V}_{e0} \times \vec{B}_0)$	$0 = -e n_{e0} (\vec{E}_0 + \vec{V}_{e0} \times \vec{B}_0)$
$m_i n_{i0} (\vec{V}_{i0} \cdot \nabla) \vec{V}_{i0}$ $= -\nabla p_{i0} + e n_{i0} (\vec{E}_0 + \vec{V}_{i0} \times \vec{B}_0)$	$0 = +e n_{i0} (\vec{E}_0 + \vec{V}_{i0} \times \vec{B}_0)$

$\left(\vec{V}_{e0} \cdot \nabla\right) \ln\left(p_{e0} n_{e0}^{-\gamma_e}\right) = 0$	$0 = 0$
$\left(\vec{V}_{i0} \cdot \nabla\right) \ln\left(p_{i0} n_{i0}^{-\gamma_i}\right) = 0$	$0 = 0$
$\nabla \cdot \vec{E}_0 = \dfrac{e(n_{i0} - n_{e0})}{\epsilon_0}$	$0 = \dfrac{e(n_{i0} - n_{e0})}{\epsilon_0}$
$\nabla \cdot \vec{B}_0 = 0$	$0 = 0$
$\nabla \times \vec{E}_0 = 0$	$0 = 0$
$\nabla \times \vec{B}_0 = \mu_0 e(n_{i0}\vec{V}_{i0} - n_{e0}\vec{V}_{e0})$	$0 = \mu_0 e(n_{i0}\vec{V}_{i0} - n_{e0}\vec{V}_{e0})$

where

$$0 = \frac{e(n_{i0} - n_{e0})}{\epsilon_0}$$

it yields

$$n_{i0} = n_{e0} = n_0 \tag{9.11}$$

Substituting Eq.(9.11) into

$$0 = \mu_0 e(n_{i0}\vec{V}_{i0} - n_{e0}\vec{V}_{e0})$$

it yields

$$0 = \mu_0 e n_0(\vec{V}_{i0} - \vec{V}_{e0})$$

That is,

$$\vec{V}_{i0} = \vec{V}_{e0} = \vec{V}_0 \quad (9.12)$$

Substituting Eq. (9.12) into the equilibrium momentum equations, it yields

$$0 = \vec{E}_0 + \vec{V}_0 \times \vec{B}_0 \quad (9.13)$$

We can choose a moving frame such that $\vec{V}_0 = 0$, it yields $\vec{E}_0 = 0$

9.3. Linearized governing equations

Let us assume the background medium is uniform. We choose a moving frame such that $\vec{V}_{e0} = \vec{V}_{i0} = 0$. Linearizing the two-fluid equations (9.1)~(9.6), it yields

$\partial n_{e1}/\partial t = -n_0 \nabla \cdot \vec{V}_{e1}$	(9.14)
$\partial n_{i1}/\partial t = -n_0 \nabla \cdot \vec{V}_{i1}$	(9.15)
$m_e n_0 (\partial \vec{V}_{e1}/\partial t) = -\nabla p_{e1} - e n_0 (\vec{E}_1 + \vec{V}_{e1} \times \vec{B}_0)$	(9.16)
$m_i n_0 (\partial \vec{V}_{i1}/\partial t) = -\nabla p_{i1} + e n_0 (\vec{E}_1 + \vec{V}_{i1} \times \vec{B}_0)$	(9.17)
$\dfrac{\partial p_{e1}}{\partial t} = \dfrac{\gamma_e p_{e0}}{n_0} \dfrac{\partial n_{e1}}{\partial t} (= -\gamma_e p_{e0} \nabla \cdot \vec{V}_{e1})$	(9.18)
$\dfrac{\partial p_{i1}}{\partial t} = \dfrac{\gamma_i p_{i0}}{n_0} \dfrac{\partial n_{i1}}{\partial t} (= -\gamma_i p_{i0} \nabla \cdot \vec{V}_{i1})$	(9.19)

Linearizing the Maxwell equations, (9.7)~(9.10), in the ion-electron two-fluid plasma, it yields

$\nabla \cdot \vec{E}_1 = \dfrac{e(n_{i1} - n_{e1})}{\epsilon_0}$	(9.20)
$\nabla \cdot \vec{B}_1 = 0$	(9.21)
$\nabla \times \vec{E}_1 = -\dfrac{\partial \vec{B}_1}{\partial t}$	(9.22)
$\nabla \times \vec{B}_1 = \mu_0 e n_0 (\vec{V}_{i1} - \vec{V}_{e1}) + \dfrac{1}{c^2}\dfrac{\partial \vec{E}_1}{\partial t}$	(9.23)

9.4. Fourier transform of the governing equations from the (t, x) domain to the (ω, k) domain

Let us consider a one-dimensional problem ($\nabla = \hat{x}\partial/\partial x$), and a uniform background magnetic field lying on the x- y plane, that is $\vec{B}_0 = B_0(\hat{x}\cos\theta + \hat{y}\sin\theta)$, where θ is the angle between the wave normal direction $\hat{k}$ and the ambient magnetic field direction. Applying the plane wave assumption

$$A_1(t, x; \theta) = Re\left\{\tilde{A}_1(\omega, k; \theta)e^{i(kx-\omega t)}\right\}$$

to the linearized equations (9.14)~(9.23) it yields

$-i\omega\tilde{n}_{e1} = -n_0(ik)\tilde{V}_{e1x}$	(9.24)
$-i\omega\tilde{n}_{i1} = -n_0(ik)\tilde{V}_{i1x}$	(9.25)

$m_e n_0(-i\omega)\tilde{\vec{V}}_{e1} = -\hat{x}(ik)\tilde{p}_{e1} - en_0(\tilde{\vec{E}}_1 + \tilde{\vec{V}}_{e1} \times \vec{B}_0)$	(9.26)
$m_i n_0(-i\omega)\tilde{\vec{V}}_{i1} = -\hat{x}(ik)\tilde{p}_{i1} + en_0(\tilde{\vec{E}}_1 + \tilde{\vec{V}}_{i1} \times \vec{B}_0)$	(9.27)
$-i\omega\tilde{p}_{e1} = -\gamma_e p_{e0}(ik)\tilde{V}_{e1x}$	(9.28)
$-i\omega\tilde{p}_{i1} = -\gamma_i p_{i0}(ik)\tilde{V}_{i1x}$	(9.29)
$ik\tilde{E}_{1x} = \dfrac{e(\tilde{n}_{i1} - \tilde{n}_{e1})}{\epsilon_0}$	(9.30)
$ik\tilde{B}_{1x} = 0$	(9.31)
$ik\hat{x} \times \tilde{\vec{E}}_1 = i\omega\tilde{\vec{B}}_1$	(9.32)
$ik\hat{x} \times \tilde{\vec{B}}_1 = \mu_0 e n_0(\tilde{\vec{V}}_{i1} - \tilde{\vec{V}}_{e1}) - \dfrac{i\omega}{c^2}\tilde{\vec{E}}_1$	(9.33)

We have discussed in Lecture 7 that waves consist of both electromagnetic component and electrostatic component can be obtained by curl of the Faraday's Law. Curl of Eq.(9.32) yields

$$ik\hat{x}\times\left(ik\hat{x}\times\tilde{\vec{E}}_1\right) = ik\hat{x}\times\left(i\omega\tilde{\vec{B}}_1\right)$$

$$= i\omega\left[\mu_0 e n_0\left(\tilde{\vec{V}}_{i1} - \tilde{\vec{V}}_{e1}\right) - \frac{i\omega}{c^2}\tilde{\vec{E}}_1\right]$$

That is

$$\left[\left(\frac{\omega^2}{c^2} - k^2\right)\vec{\vec{1}} + k^2\hat{x}\hat{x}\right]\cdot\ \tilde{\vec{E}}_1 = -i\omega\mu_0 e n_0\left(\tilde{\vec{V}}_{i1} - \tilde{\vec{V}}_{e1}\right) \qquad (9.34)$$

If we can obtain $\tilde{\vec{V}}_{i1}$ and $\tilde{\vec{V}}_{e1}$ as function of $\tilde{\vec{E}}_1$, then we can obtain an equation of $\tilde{\vec{E}}_1$ from equation (9.34).

Substituting Eqs. (9.28) and (9.29) into Eqs. (9.26) and (9.27) respectively, and multiplying the resulting equations by $i\omega/k^2 m_\alpha n_0$, where $\alpha = i$ or e, it yields

$$\frac{\omega^2}{k^2}\tilde{\vec{V}}_{e1} = \frac{\gamma_e p_{e0}}{m_e n_0}\hat{x}\hat{x}\cdot\tilde{\vec{V}}_{e1} - \frac{i\omega}{k^2}\frac{e}{m_e}(\tilde{\vec{E}}_1 + \tilde{\vec{V}}_{e1}\times\vec{B}_0)$$	(9.35)
$$\frac{\omega^2}{k^2}\tilde{\vec{V}}_{i1} = \frac{\gamma_i p_{i0}}{m_i n_0}\hat{x}\hat{x}\cdot\tilde{\vec{V}}_{i1} + \frac{i\omega}{k^2}\frac{e}{m_i}(\tilde{\vec{E}}_1 + \tilde{\vec{V}}_{i1}\times\vec{B}_0)$$	(9.36)

Eqs. (9.35) and (9.36) can be rewritten in the following forms

$$\vec{\vec{M}}_e^{-1}\cdot\tilde{\vec{V}}_{e1} = \tilde{\vec{E}}_1$$	(9.37)
$$\vec{\vec{M}}_i^{-1}\cdot\tilde{\vec{V}}_{i1} = \tilde{\vec{E}}_1$$	(9.38)

Exercise 9.1:

Find the inverse of the mobility tensors $\vec{\vec{M}}_e^{-1}$ and $\vec{\vec{M}}_i^{-1}$

Eqs. (9.37) and (9.38) yields

$\tilde{\vec{V}}_{e1} = \vec{\vec{M}}_e \cdot \tilde{\vec{E}}_1$	(9.39)
$\tilde{\vec{V}}_{i1} = \vec{\vec{M}}_i \cdot \tilde{\vec{E}}_1$	(9.40)

Substituting Eqs. (9.39) and (9.40) into Eq. (9.34) to eliminate the $\tilde{\vec{V}}_{i1}$ and $\tilde{\vec{V}}_{e1}$, it yields

$\left[\left(\frac{\omega^2}{c^2} - k^2\right)\vec{\vec{1}} + k^2\hat{x}\hat{x} + i\omega\mu_0 e n_0\left(\vec{\vec{M}}_i - \vec{\vec{M}}_e\right)\right] \cdot \tilde{\vec{E}}_1 = 0$	(9.41)

Let

$\vec{\vec{D}} = \left(1 - \frac{c^2k^2}{\omega^2}\right)\vec{\vec{1}} + \frac{c^2k^2}{\omega^2}\hat{x}\hat{x} + \frac{i\omega e n_0}{\omega^2\epsilon_0}\left(\vec{\vec{M}}_i - \vec{\vec{M}}_e\right)$	(9.42)

Eq. (9.41) can be rewritten as

$(\omega^2/c^2)\vec{\vec{D}} \cdot \tilde{\vec{E}}_1 = 0$	(9.43)

Exercise 9.2: Find the wave dielectric tensor $\overleftrightarrow{D}(\omega, k; \theta)$

For $\tilde{\overleftrightarrow{E}}_1 \neq 0$, Eq. (9.43) implies $\det\left(\overleftrightarrow{D}\right) = 0$. The eigen-mode solutions $\omega(k;\theta)$ are also called the dispersion relations of the ion-electron two-fluid plasma. Examples of the two-fluid dispersion relations of are given in Figures 9.1 and 9.2, where $\omega_{UH} = \sqrt{\omega_{pe}^2 + \Omega_e^2}$ is the upper hybrid frequency and $\omega_{LH} = \sqrt{\Omega_e \Omega_i}$ is the lower hybrid frequency.

There are the six propagating wave modes in the ion-electron two-fluid plasma, where the high frequency waves, the ion acoustic waves, and the very low frequency MHD waves can be obtained from a set of simplified equations. We will discuss the high frequency waves and the ion acoustic waves in this Lecture, and discuss the MHD waves in Lecture 10.

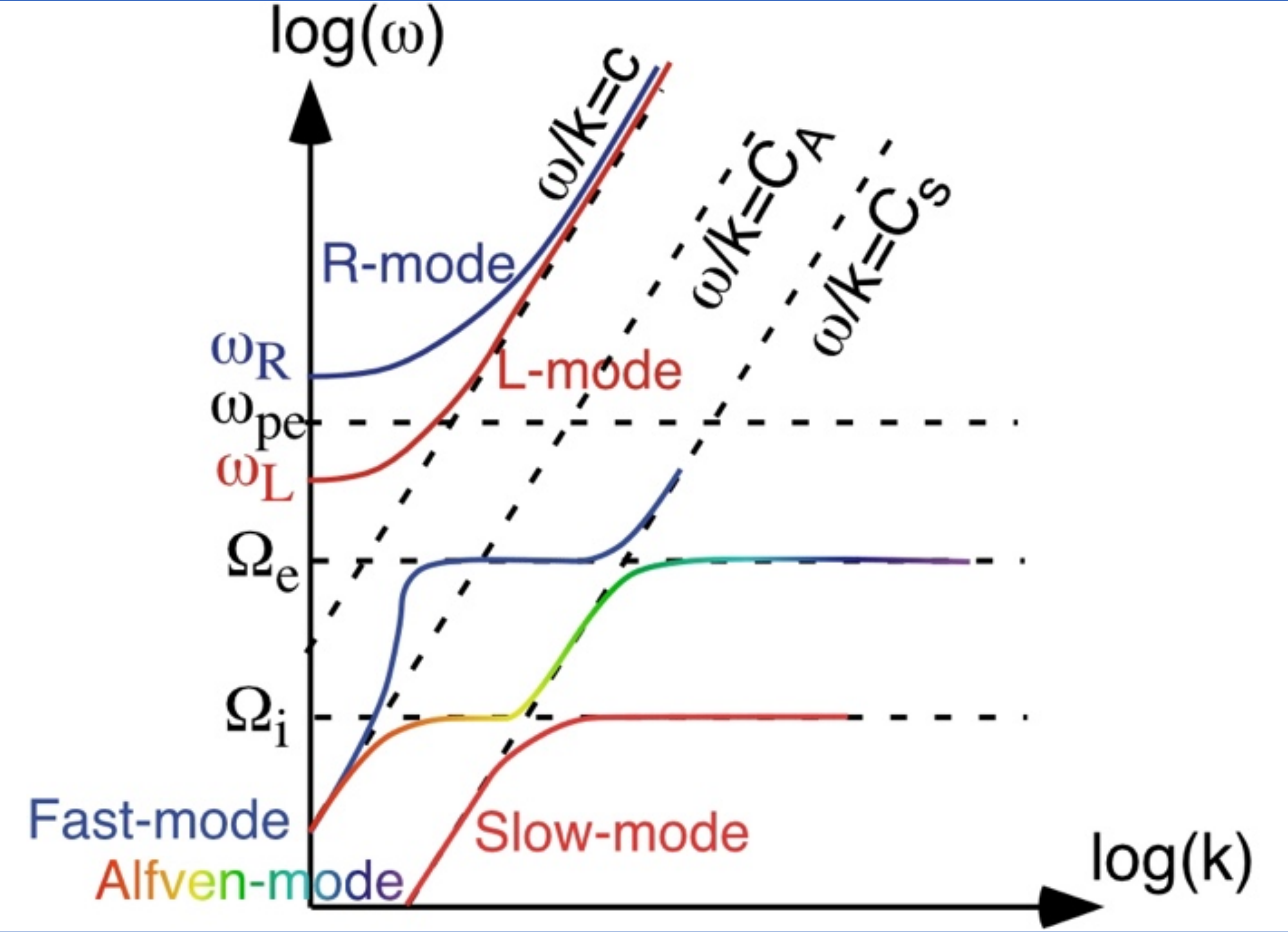

Figure 9.1. A sketch of the dispersion relation of five parallel propagating wave modes in the ion-electron two-fluid plasma. Langmuir wave does not includ in this plot.

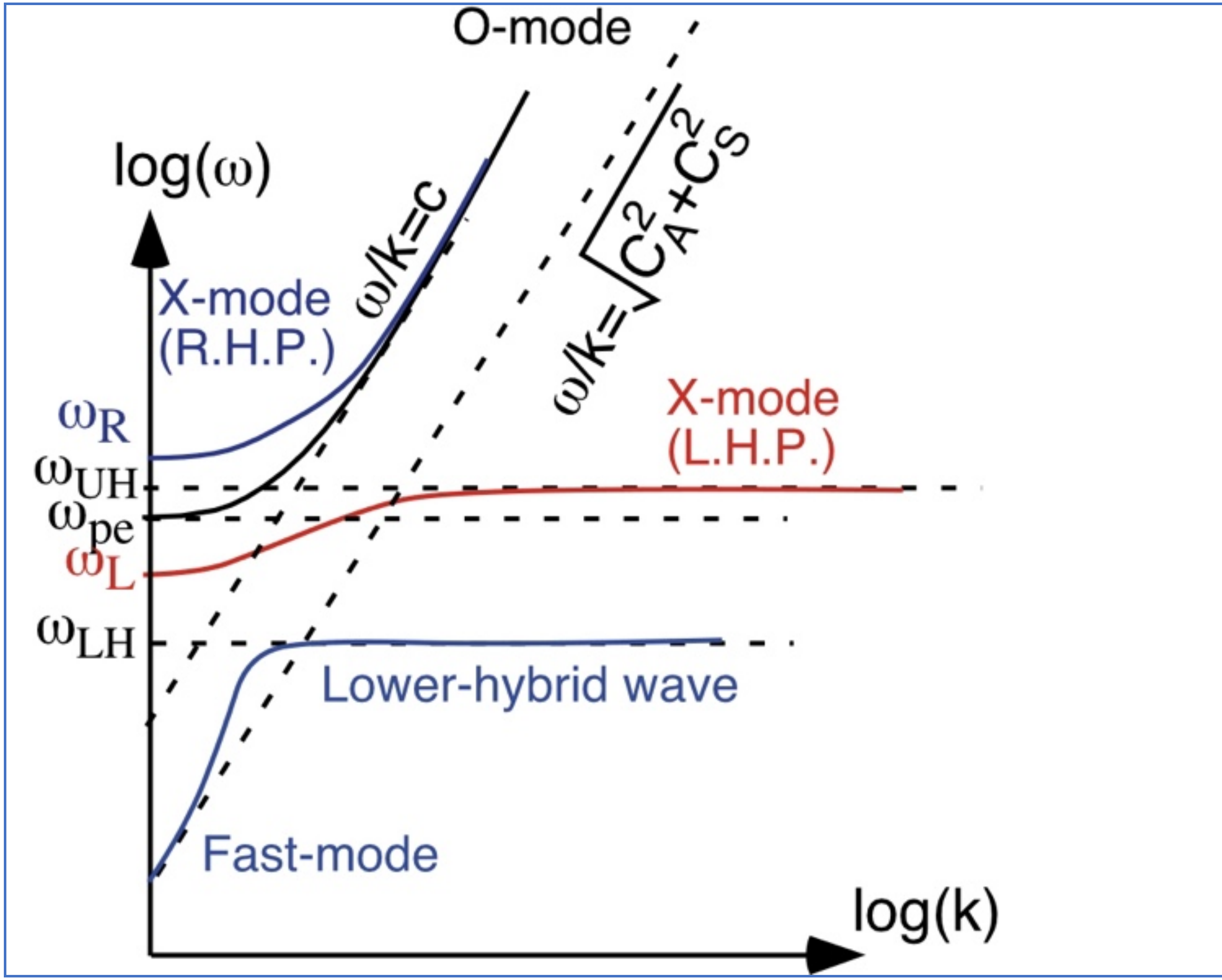

Figure 9.2. A sketch of the dispersion relation of four perpendicular propagating wave modes in the ion-electron two-fluid plasma.

9.5. High-frequency waves in an un-magnetized plasma ($\vec{B}_0 = 0$)

To study waves with frequency equal or higher than the electrons' characteristic frequencies, we can assume that ions will not respond to the waves. Namely, $n_{i1} = 0$, $p_{i1} = 0$, $\vec{V}_{i1} = 0$. For un-magnetized plasma $\vec{B}_0 = 0$, the governing equations in the (k, ω) domain can be obtained from Eqs. (9.24)~(9.33), i.e.,

$-i\omega\tilde{n}_{e1} = -n_0(ik)\tilde{V}_{e1x}$	(9.44)
$m_e n_0(-i\omega)\tilde{\vec{V}}_{e1} = -\hat{x}(ik)\tilde{p}_{e1} - en_0\tilde{\vec{E}}_1$	(9.45)
$-i\omega\tilde{p}_{e1} = -\gamma_e p_{e0}(ik)\tilde{V}_{e1x}$	(9.46)

$$ik\tilde{E}_{1x} = \frac{e(-\tilde{n}_{e1})}{\epsilon_0}$$	(9.47)
$$ik\tilde{B}_{1x} = 0$$	(9.48)
$$ik\hat{x}\times\tilde{\vec{E}}_1 = i\omega\tilde{\vec{B}}_1$$	(9.49)
$$ik\hat{x}\times\tilde{\vec{B}}_1 = \mu_0 e n_0(-\tilde{\vec{V}}_{e1}) - \frac{i\omega}{c^2}\tilde{\vec{E}}_1$$	(9.50)

Substituting (9.46) into (9.45) to eliminate $\tilde{p}_{e1}$ and then substituting the resulting equation into equation (9.50) to eliminate $\tilde{\vec{V}}_{e1}$, and substituting (9.49) into (9.50) to eliminate $\tilde{\vec{B}}_1$, it yields $\vec{\vec{D}} \cdot \tilde{\vec{E}}_1 = 0$. i.e.,

$$\begin{bmatrix} D_{xx} & 0 & 0 \\ 0 & D_{yy} & 0 \\ 0 & 0 & D_{zz} \end{bmatrix} \cdot \begin{bmatrix} \tilde{E}_{1x} \\ \tilde{E}_{1y} \\ \tilde{E}_{1z} \end{bmatrix} = 0$$

where

$$D_{xx} = 1 - \frac{\omega_{pe0}^2}{\omega^2 - C_{e0}^2 k^2}$$	
$$D_{yy} = D_{zz} = 1 - \frac{c^2 k^2}{\omega^2} - \frac{\omega_{pe0}^2}{\omega^2}$$	

where $\omega_{pe0}^2 = n_0 e^2 / m_e \epsilon_0$ and $C_{e0}^2 = \gamma_e p_{e0} / n_0 m_e$.

For $\tilde{E}_{1x} \neq 0$, but $\tilde{E}_{1y} = \tilde{E}_{1z} = 0$, it yields $D_{xx} = 0$. i.e.,

$$\omega^2 = \omega_{pe0}^2 + C_{e0}^2 k^2 \tag{9.51}$$

Eq. (9.51) is the Langmuir wave dispersion relation. Since $\tilde{E}_{1x} \neq 0$ implies $\tilde{n}_{e1} \neq 0$ and $\tilde{V}_{1x} \neq 0$, the Langmuir wave is an electrostatic wave, a compressional wave, and a longitudinal wave.

For $\tilde{E}_{1x} = 0$, but $(\tilde{E}_{1y}, \tilde{E}_{1z}) \neq (0,0)$, it yields $D_{yy} = D_{zz} = 0$. i.e.,

$$\omega^2 = \omega_{pe0}^2 + c^2 k^2 \tag{9.52}$$

Eq. (9.52) is the light wave dispersion relation. Since $\tilde{E}_{1x} = 0$ implies $\tilde{n}_{e1} = 0$ and $\tilde{V}_{1x} = 0$, the light wave is an electromagnetic incompressible pure transverse wave with a cut-off frequency at $\omega = \omega_{pe0}$.

Figure 9.3 is a sketch of the (a) Langmuir wave and (b) light waves propagating in an un-magnetized plasma. The wave frequency at $k = 0$ is the cut-off frequency. Both of the wave modes have a cut-off frequency at $\omega = \omega_{pe}$. The waves can only propagate at the frequency $\omega > \omega_{pe}$. No wave can propagate at frequency below the cut-off frequency. It is because that, for $\omega < \omega_{pe}$, we have $k^2 < 0$, which implies that the wave amplitude will decrease exponentially and become a non-propagating wave.

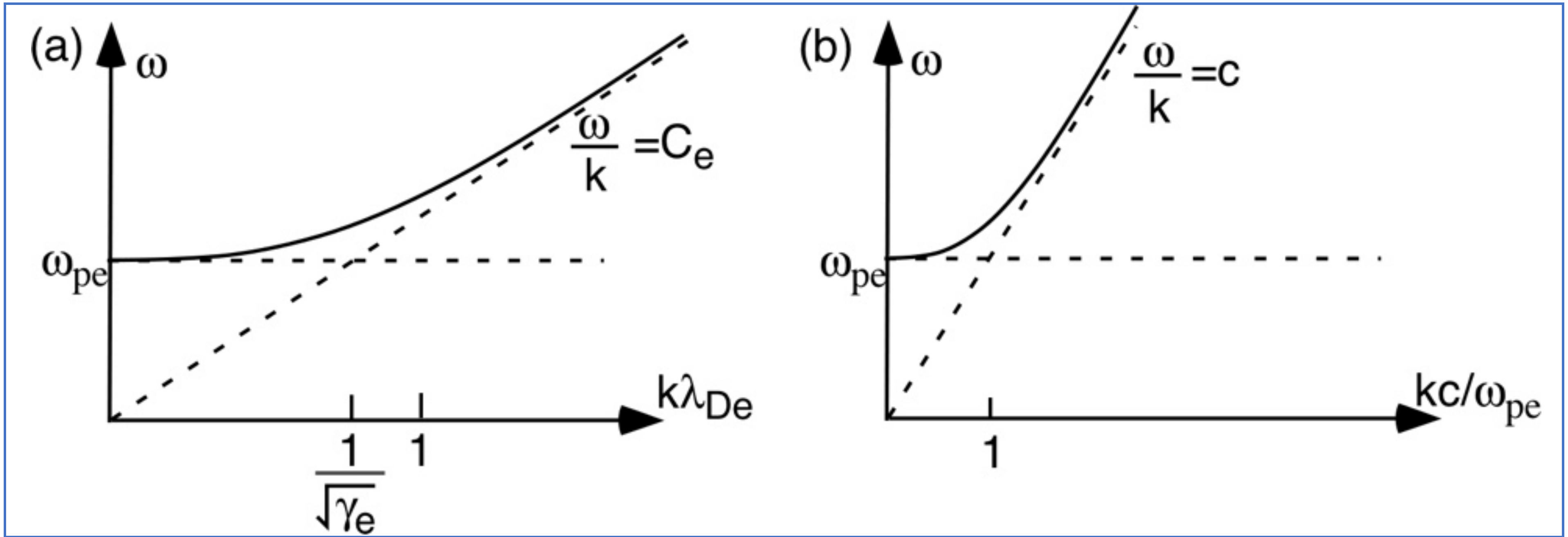

Figure 9.3. A Sketch of the (a) Langmuir wave and (b) light waves propagating in an un-magnetized plasma.

9.6. High-frequency waves in a magnetized plasma ($B_0 > 0$)

To study waves with frequency equal or higher than the electrons' characteristic frequencies, we can assume that ions will not respond to the waves. i.e., $n_{i1} = 0$, $p_{i1} = 0$, $\vec{V}_{i1} = 0$. Thus, the wave dielectric tensor can be rewritten as

$$\overset{\rightrightarrows}{D}(\omega, k; \theta) = \left(1 - \frac{c^2k^2}{\omega^2}\right)\overset{\rightrightarrows}{1} + \frac{c^2k^2}{\omega^2}\hat{x}\hat{x} - \frac{i\omega e n_0}{\omega^2 \epsilon_0}\overset{\rightrightarrows}{M}_e$$

or

$$\overset{\rightrightarrows}{D}(\omega, k; \theta) = \left(1 - \frac{c^2k^2}{\omega^2}\right)\overset{\rightrightarrows}{1} + \frac{c^2k^2}{\omega^2}\hat{x}\hat{x} - i\omega\frac{\omega_{pe0}^2}{\omega^2}\frac{m_e}{e}\overset{\rightrightarrows}{M}_e$$

9.6.1. Parallel propagating high-frequency waves ($\vec{k} \parallel \vec{B}_0$)

To study waves with frequency equal or higher than the electrons' characteristic frequencies, we can assume that ions will not respond to the waves. i.e., $n_{i1} = 0$, $p_{i1} = 0$, $\vec{V}_{i1} = 0$.

For parallel propagating waves, let $\vec{k} = \hat{x}k$, $\vec{B}_0 = \hat{x}B_0$. The governing equations in the (k, ω) domain can be obtained from Eqs. (9.24)~(9.33), i.e.,

$-i\omega\tilde{n}_{e1} = -n_0(ik)\tilde{V}_{e1x}$	(9.53)
$m_e n_0(-i\omega)\tilde{\vec{V}}_{e1} = -\hat{x}(ik)\tilde{p}_{e1} - en_0(\tilde{\vec{E}}_1 + \tilde{\vec{V}}_{e1}\times\hat{x}B_0)$	(9.54)
$-i\omega\tilde{p}_{e1} = -\gamma_e p_{e0}(ik)\tilde{V}_{e1x}$	(9.55)

$ik\tilde{E}_{1x} = \dfrac{e(-\tilde{n}_{e1})}{\epsilon_0}$	(9.56)
$ik\tilde{B}_{1x} = 0$	(9.57)
$ik\hat{x}\times\overset{\rightrightarrows}{\tilde{E}}_1 = i\omega\overset{\rightrightarrows}{\tilde{B}}_1$	(9.58)
$ik\hat{x}\times\overset{\rightrightarrows}{\tilde{B}}_1 = \mu_0 e n_0(-\overset{\rightrightarrows}{\tilde{V}}_{e1}) - \dfrac{i\omega}{c^2}\overset{\rightrightarrows}{\tilde{E}}_1$	(9.59)

Substituting (9.55) into (9.54) to eliminate $\tilde{p}_{e1}$ and then substituting the resulting equation into equation (9.59) to eliminate $\overset{\rightrightarrows}{\tilde{V}}_{e1}$, and substituting (9.58) into (9.59) to eliminate $\overset{\rightrightarrows}{\tilde{B}}_1$, it yields $\overset{\rightrightarrows}{D}\cdot\overset{\rightrightarrows}{\tilde{E}}_1 = 0$. i.e.,

$$\begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & iD_6 \\ 0 & -iD_6 & D_3 \end{bmatrix} \cdot \begin{bmatrix} \tilde{E}_{1x} \\ \tilde{E}_{1y} \\ \tilde{E}_{1z} \end{bmatrix} = 0$$

where

$$D_1 = 1 - \frac{\omega_{pe0}^2}{\omega^2 - C_{e0}^2 k^2}$$	
$$D_2 = D_3 = 1 - \frac{c^2 k^2}{\omega^2} - \frac{\omega_{pe0}^2}{\omega^2 - \Omega_{e0}^2}$$	
$$D_6 = \frac{\omega_{pe0}^2}{\omega^2 - \Omega_{e0}^2} \frac{\Omega_{e0}}{\omega}$$	

where $\omega_{pe0}^2 = n_0 e^2 / m_e \epsilon_0$, $C_{e0}^2 = \gamma_e p_{e0} / n_0 m_e$, and $\Omega_{e0} = eB_0 / m_e$.

For $\tilde{E}_{1x} \neq 0$, but $\tilde{E}_{1y} = \tilde{E}_{1z} = 0$, it yields $D_1 = 0$. i.e.,

$$\omega^2 = \omega_{pe0}^2 + C_{e0}^2 k^2$$

This is the dispersion relation of the Langmuir wave, which is the same Langmuir wave obtained in un-magnetized plasma. Since $\tilde{E}_{1x} \neq 0$ implies $\tilde{n}_{e1} \neq 0$ and $\tilde{V}_{1x} \neq 0$, the Langmuir wave is an electrostatic compressional longitudinal wave.

For $\tilde{E}_{1x} = 0$, but $(\tilde{E}_{1y}, \tilde{E}_{1z}) \neq (0,0)$, it yields

$$\det\begin{bmatrix} D_2 & iD_6 \\ -iD_6 & D_2 \end{bmatrix} = D_2^2 - D_6^2 = (D_2 + D_6)(D_2 - D_6) = 0$$

Namely, we have dispersion relation of two modes. One is for $D_2 + D_6 = 0$, the other is for $D_2 - D_6 = 0$. Since $\tilde{E}_{1x} = 0$ implies $\tilde{n}_{e1} = 0$ and $\tilde{V}_{1x} = 0$, both wave-mode are electromagnetic incompressible transverse waves.

For $D_2 + D_6 = 0$, it yields

$$1 - \frac{c^2k^2}{\omega^2} - \frac{\omega_{pe0}^2}{\omega(\omega + \Omega_{e0})} = 0$$

or

$$\omega^2 = c^2k^2 + \omega_{pe0}^2 \frac{\omega}{\omega + \Omega_{e0}} \tag{9.60}$$

Since

$$\begin{bmatrix} D_2 & iD_6 \\ -iD_6 & D_2 \end{bmatrix} \begin{bmatrix} \tilde{E}_{1y} \\ \tilde{E}_{1z} \end{bmatrix} = 0$$

$D_2 + D_6 = 0$ yields $\tilde{E}_{1y} = i\tilde{E}_{1z}$. It can be shown that the wave electric field is left-hand polarized w.r.t. the ambient magnetic field ($\vec{B}_0 = \hat{x}B_0$). Thus, Eq. (9.60) is the dispersion relation of the L-mode wave.

At $k = 0$, Eq. (9.60) yields

$$\omega_{\text{cut-off}}^2 + \Omega_{e0}\omega_{\text{cut-off}} - \omega_{pe0}^2 = 0$$

The positive solution of $\omega_{\text{cut-off}}$ is

$$\omega_{\text{cut-off}} = \frac{1}{2}(-\Omega_{e0} + \sqrt{\Omega_{e0}^2 + 4\omega_{pe0}^2})$$

Thus, the cut-off frequency of the L-mode is

$$\omega_{\text{cut-off,L-mode}} = \omega_L = \frac{1}{2}(-\Omega_{e0} + \sqrt{\Omega_{e0}^2 + 4\omega_{pe0}^2})$$

For $\omega_{pe0}^2 \gg \Omega_{e0}^2$, the cut-off frequency of the L-mode is approximately

$$\omega_L \approx \omega_{pe0} - \frac{\Omega_{e0}}{2}(1 - \frac{\Omega_{e0}}{4\omega_{pe0}})$$

which is slightly less than the cut-off frequency of the EM wave propagating in the un-magnetized plasma.

For $D_2 - D_6 = 0$, it yields

$$1 - \frac{c^2k^2}{\omega^2} - \frac{\omega_{pe0}^2}{\omega(\omega - \Omega_{e0})} = 0$$

or

$$\omega^2 = c^2k^2 + \omega_{pe0}^2 \frac{\omega}{\omega - \Omega_{e0}} \tag{9.61}$$

Since

$$\begin{bmatrix} D_2 & iD_6 \\ -iD_6 & D_2 \end{bmatrix} \begin{bmatrix} \tilde{E}_{1y} \\ \tilde{E}_{1z} \end{bmatrix} = 0$$

$D_2 - D_6 = 0$ yields $\tilde{E}_{1y} = -i\tilde{E}_{1z}$. It can be shown that the wave electric field is right-hand polarized w.r.t. the ambient magnetic field ($\vec{B}_0 = \hat{x}B_0$). Thus, Eq. (9.61) is the dispersion relation of the R-mode wave.

At $k = 0$, Eq. (9.61) yields

$$\omega_{\text{cut-off}}^2 - \Omega_{e0}\omega_{\text{cut-off}} - \omega_{pe0}^2 = 0$$

The positive solution of $\omega_{\text{cut-off}}$ is

$$\omega_{\text{cut-off}} = \frac{1}{2}(\Omega_{e0} + \sqrt{\Omega_{e0}^2 + 4\omega_{pe0}^2})$$

Thus, the cut-off frequency of the R-mode is

$$\omega_{\text{cut-off,R-mode}} = \omega_R = \frac{1}{2}(\Omega_{e0} + \sqrt{\Omega_{e0}^2 + 4\omega_{pe0}^2})$$

For $\omega_{pe0}^2 \gg \Omega_{e0}^2$, the cut-off frequency of the R-mode is approximately

$$\omega_R \approx \omega_{pe0} + \frac{\Omega_{e0}}{2}(1 + \frac{\Omega_{e0}}{4\omega_{pe0}})$$

which is slightly greater than the cut-off frequency of the EM wave propagating in the un-magnetized plasma.

Dispersion relations of the parallel propagating high-frequency waves are sketched in Figure 9.4.

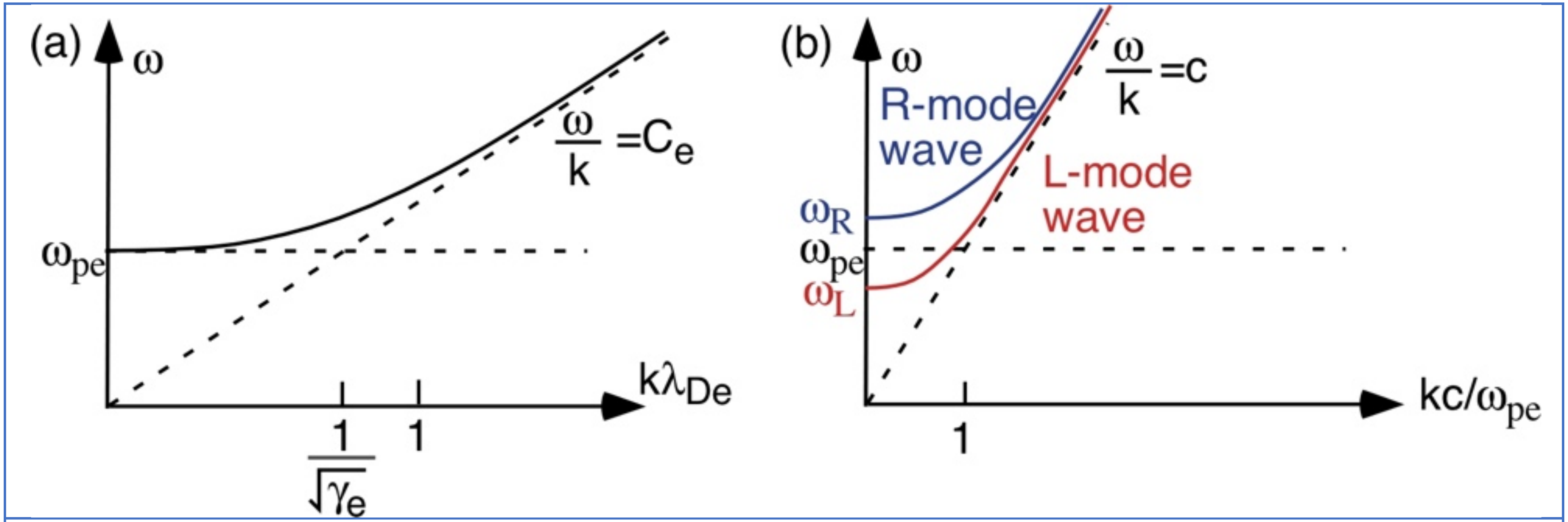

Figure 9.4. A Sketch of the dispersion relations of the parallel propagating high-frequency waves in the ion-electron two-fluid plasma.

9.6.2. Faraday rotation and it applications

Figure 9.4(b) shows the dispersion relation of the R-mode and L-mode, where

R-mode dispersion relation

$$\omega^2 = c^2k^2 + \omega_{pe0}^2 \frac{\omega}{\omega - \Omega_{e0}}$$

L-mode dispersion relation

$$\omega^2 = c^2k^2 + \omega_{pe0}^2 \frac{\omega}{\omega + \Omega_{e0}}$$

It can be shown that for a given ω the R-mode has smaller wave number k, that is the L-mode wave has a shorter wavelength. Thus, at a distance from the source region the received L-mode phase will be ahead of the R-mode. Thus, the polarization plane of the EM wave will rotate left-handed w.r.t. the ambient magnetic field. The left-hand rotation of the polarization plane is called the "**Faraday rotation**."

Application:

Scientists estimate the line-of-sight magnetic field strength on the solar surface or on the surface of a distant star based on the strength of the observed Zeeman effect.

Scientists estimate the direction of the line-of-sight magnetic field on the solar surface or the surface of a distant star based on the theory of Faraday rotation. 也就是比較兩片相反方向旋轉的光柵所收到的光線強度差異，來決定光線極化方向如何旋轉，進一步決定恆星表面磁場方向。

9.6.3. Perpendicular propagating waves ($\vec{k} \perp \vec{B}_0$)

To study waves with frequency equal or higher than the electrons' characteristic frequencies, we can assume that ions will not respond to the waves. i.e., $n_{i1} = 0$, $p_{i1} = 0$, $\vec{V}_{i1} = 0$.

Let $\vec{k} = \hat{x}k$, $\vec{B}_0 = \hat{y}B_0$. The governing equations in the (k, ω) domain can be obtained from Eqs. (9.24)~(9.33), i.e.,

$-i\omega\tilde{n}_{e1} = -n_0(ik)\tilde{V}_{e1x}$	(9.62)
$m_e n_0(-i\omega)\tilde{\vec{V}}_{e1} = -\hat{x}(ik)\tilde{p}_{e1} - en_0(\tilde{\vec{E}}_1 + \tilde{\vec{V}}_{e1}\times\hat{y}B_0)$	(9.63)
$-i\omega\tilde{p}_{e1} = -\gamma_e p_{e0}(ik)\tilde{V}_{e1x}$	(9.64)

$ik\tilde{E}_{1x} = \frac{e(-\tilde{n}_{e1})}{\epsilon_0}$	(9.65)
$ik\tilde{B}_{1x} = 0$	(9.66)
$ik\hat{x}\times\tilde{\vec{\vec{E}}}_1 = i\omega\tilde{\vec{\vec{B}}}_1$	(9.67)
$ik\hat{x}\times\tilde{\vec{\vec{B}}}_1 = \mu_0 e n_0(-\tilde{\vec{\vec{V}}}_{e1}) - \frac{i\omega}{c^2}\tilde{\vec{\vec{E}}}_1$	(9.68)

Substituting (9.64) into (9.63) to eliminate $\tilde{p}_{e1}$ and then substituting the resulting equation into equation (9.68) to eliminate $\tilde{\vec{\vec{V}}}_{e1}$, and substituting (9.67) into (9.68) to eliminate $\tilde{\vec{\vec{B}}}_1$, it yields $\vec{\vec{D}} \cdot \tilde{\vec{\vec{E}}}_1 = 0$. i.e.,

$$\begin{bmatrix} D_1 & 0 & -iD_5 \\ 0 & D_2 & 0 \\ iD_5 & 0 & D_3 \end{bmatrix} \cdot \begin{bmatrix} \tilde{E}_{1x} \\ \tilde{E}_{1y} \\ \tilde{E}_{1z} \end{bmatrix} = 0$$

where

$D_1 = 1 - \dfrac{\omega_{pe0}^2}{\omega^2 - C_{e0}^2 k^2 - \Omega_{e0}^2}$	
$D_2 = 1 - \dfrac{c^2 k^2}{\omega^2} - \dfrac{\omega_{pe0}^2}{\omega^2}$	
$D_3 = 1 - \dfrac{c^2 k^2}{\omega^2} - \dfrac{\omega_{pe0}^2}{\omega^2} \dfrac{\omega^2 - C_{e0}^2 k^2}{\omega^2 - C_{e0}^2 k^2 - \Omega_{e0}^2}$	
$D_5 = \dfrac{\omega_{pe0}^2}{\omega^2 - C_{e0}^2 k^2 - \Omega_{e0}^2} \dfrac{\Omega_{e0}}{\omega}$	

For $\tilde{E}_{1y} \neq 0$, but $\tilde{E}_{1x} = \tilde{E}_{1z} = 0$, it yields $D_2 = 0$. i.e.,

$$\omega^2 = \omega_{pe0}^2 + c^2 k^2$$

This dispersion relation is the same as the electromagnetic waves obtained in the un-magnetized plasma wave. Thus, it is called the ordinary wave mode. Or simply, the O-mode wave. Since $\tilde{E}_{1x} = 0$ implies $\tilde{n}_{e1} = 0$ and $\tilde{V}_{1x} = 0$, the O-mode wave is an electromagnetic, incompressible, pure transverse wave.

For $\tilde{E}_{1y} = 0$, but $(\tilde{E}_{1x}, \tilde{E}_{1z}) \neq (0,0)$, it yields

$$\det\begin{bmatrix} D_1 & -iD_5 \\ iD_5 & D_3 \end{bmatrix} = D_1 D_3 - D_5^2 = 0$$

This dispersion relation has two wave modes solutions, both of them are hybrid waves, which consist of electromagnetic and electrostatic wave components.

Dispersion relations of the perpendicular propagating high-frequency waves are sketched in Figure 9.5.

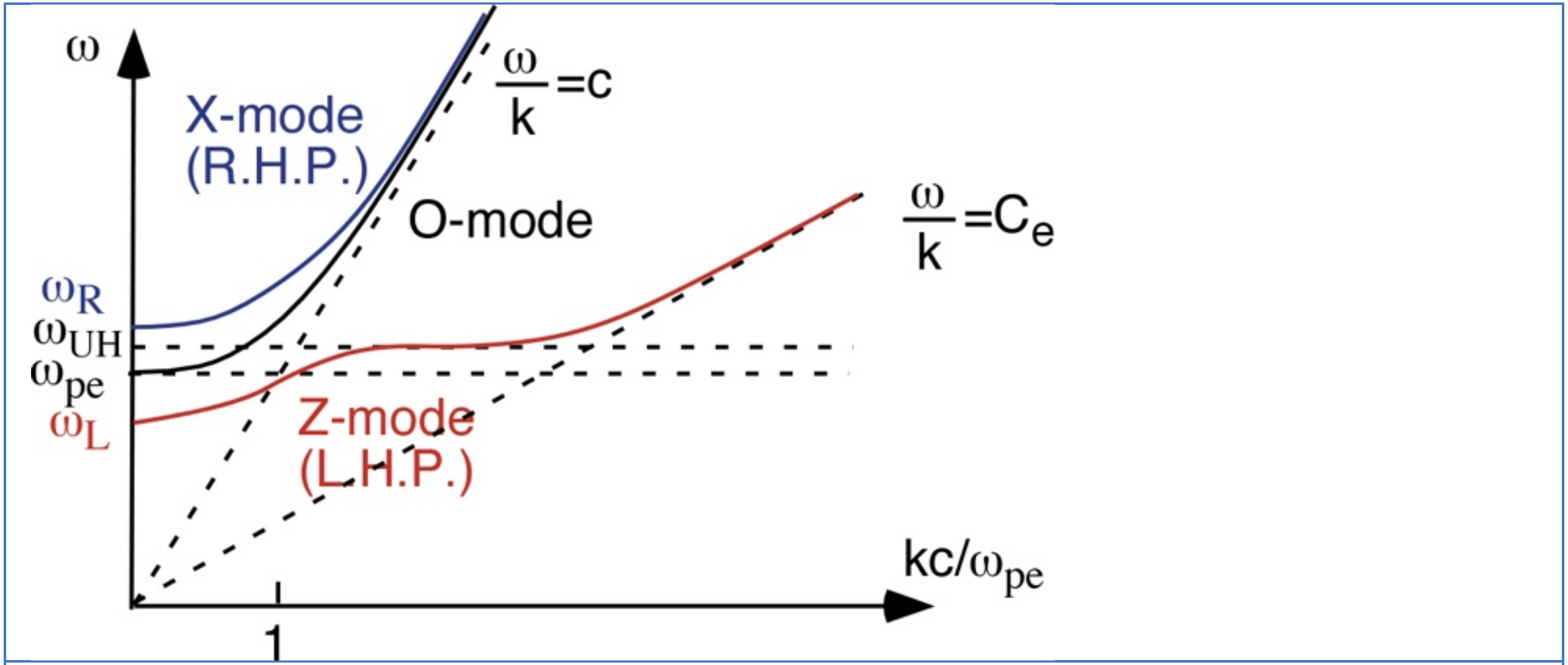

Figure 9.5. A Sketch of the dispersion relations of the perpendicular propagating high-frequency waves in the ion-electron two-fluid plasma.

為什麼會形成含有 ES＆EM 分量的混合波(hybrid wave) 呢？

假想最初是一個靜電的縱波擾動：$\tilde{E}_{1x} \neq 0$, but $\tilde{E}_{1y} = \tilde{E}_{1z} = 0$

由 Eq. (9.65) 推得若$\tilde{E}_{1x} \neq 0$, 則 $\tilde{n}_{e1} \neq 0$.

由 Eq. (9.62) 推得若$\tilde{n}_{e1} \neq 0$, 則 $\tilde{V}_{e1x} \neq 0$.

由 Eq. (9.64) 推得若$\tilde{V}_{e1x} \neq 0$, 則 $\tilde{p}_{e1} \neq 0$.

由 Eq. (9.63) 推得若$\tilde{V}_{e1x} \neq 0$, 則$\left(\tilde{\vec{V}}_{e1} \times \hat{y} B_0\right)_z \neq 0$, 則$\tilde{V}_{e1z} \neq 0$.

由 Eq. (9.67) 推得若$\tilde{V}_{e1z} \neq 0$, 則 $\tilde{E}_{1z} \neq 0$ & $\tilde{B}_{1y} \neq 0$.

因此當 $\vec{k} \perp \vec{B}_0$ 時，靜電的縱波擾動，會進一步產生電磁波的橫波擾動。

假想最初是一個電磁的橫波擾動且擾動電場垂直背景磁場：

$\tilde{E}_{1z} \neq 0$, but $\tilde{E}_{1x} = \tilde{E}_{1y} = 0$

由 Eq. (9.67) 推得若$\tilde{E}_{1z} \neq 0$, 則 $\tilde{B}_{1y} \neq 0$.

由 Eq. (9.63) 推得若$\tilde{E}_{1z} \neq 0$, 則 $\tilde{V}_{e1z} \neq 0$.

由 Eq. (9.63) 推得若$\tilde{V}_{e1z} \neq 0$, 則$\left(\tilde{\vec{V}}_{e1}\times\hat{y}B_0\right)_x \neq 0$, 則$\tilde{V}_{e1x} \neq 0$.

由 Eq. (9.62) 推得若$\tilde{V}_{e1x} \neq 0$, 則 $\tilde{n}_{e1} \neq 0$.

由 Eq. (9.64) 推得若$\tilde{V}_{e1x} \neq 0$, 則 $\tilde{p}_{e1} \neq 0$.

由 Eq. (9.65) 推得若 $\tilde{n}_{e1} \neq 0$, 則 $\tilde{E}_{1x} \neq 0$.

因此當 $\vec{k} \perp \vec{B}_0$ 時，電磁波的橫波擾動，會進一步產生靜電的縱波擾動。

只有當一個電磁的橫波擾動且擾動電場平行背景磁場時，此擾動才能維持其橫波特性，不會產生縱波分量。例如，

假想最初是一個電磁的橫波擾動且擾動電場平行背景磁場：

$\tilde{E}_{1y} \neq 0$, but $\tilde{E}_{1x} = \tilde{E}_{1z} = 0$

由 Eq. (9.67) 推得若$\tilde{E}_{1y} \neq 0$, 則 $\tilde{B}_{1z} \neq 0$.

由 Eq. (9.63) 推得若$\tilde{E}_{1y} \neq 0$, 則 $\tilde{V}_{e1y} \neq 0$.

由 Eq. (9.63) 推得若$\tilde{V}_{e1y} \neq 0$, 則 $\tilde{\vec{V}}_{e1} \times \hat{y} B_0 = 0$, 因此不會產生其他分量的電漿流場（平均速度場）擾動。於是，整個系統，都將只有 $\tilde{E}_{1y}$, $\tilde{B}_{1z}$, & $\tilde{V}_{e1y}$ 的擾動，是一個純粹的橫波擾動。此波的頻散關係與非磁化電漿中的電磁波相同，故稱為 O-mode.

反之，前面兩例，所獲得的兩種波模，都是混合波，故稱為 extraordinary waves. 這兩種混合波：一個是右旋的波，有時稱作 RX-mode，也常簡稱為 X-mode；一個是左旋的的波，有時稱作 LX-mode，也常簡稱為 Z-mode。其中，右旋的 X-mode 波動的角頻率都在ω_{pe0}之上， 擾動電場垂直 $\hat{k}$ 的電磁分量，大於平行$\hat{k}$ 的靜電分量。反之，左旋的 Z-mode 波動的角頻率有一部分低於ω_{pe0}，因此 ions 靜止不動的假設，有時會造成錯誤的結果。此外，Z-mode 波動的擾動電場平行$\hat{k}$ 的靜電分量，通常大於垂直 $\hat{k}$ 的電磁分量，這也是區分 Z-mode 與 X-mode 的方法之一。

It can be shown that the cut-off frequency of the right-hand polarized X-mode wave is the same as the cut-off frequency of the parallel propagating R-mode wave, ω_R, which indeed is also the cut-off frequency of the all high-frequency right-hand polarized waves that propagating oblique to the ambient magnetic field.

Likewise, the cut-off frequency of the left-hand polarized Z-mode wave is the same as the cut-off frequency of the parallel propagating L-mode wave, ω_L, which indeed is also the cut-off frequency of the all high-frequency left-hand polarized waves that propagating oblique to the ambient magnetic field.

9.7. Linear wave solutions in an un-magnetized plasma ($B_0 = 0$)

$-i\omega\tilde{n}_{e1} = -n_0(ik)\tilde{V}_{e1x}$	(9.69)
$-i\omega\tilde{n}_{i1} = -n_0(ik)\tilde{V}_{i1x}$	(9.70)
$m_e n_0(-i\omega)\tilde{\vec{V}}_{e1} = -\hat{x}(ik)\tilde{p}_{e1} - en_0\tilde{\vec{E}}_1$	(9.71)
$m_i n_0(-i\omega)\tilde{\vec{V}}_{i1} = -\hat{x}(ik)\tilde{p}_{i1} + en_0\tilde{\vec{E}}_1$	(9.72)
$-i\omega\tilde{p}_{e1} = -\gamma_e p_{e0}(ik)\tilde{V}_{e1x}$	(9.73)
$-i\omega\tilde{p}_{i1} = -\gamma_i p_{i0}(ik)\tilde{V}_{i1x}$	(9.74)
$ik\tilde{E}_{1x} = e(\tilde{n}_{i1} - \tilde{n}_{e1})/\epsilon_0$	(9.75)
$ik\tilde{B}_{1x} = 0$	(9.76)

$ik\hat{x}\times\tilde{\vec{E}}_1 = i\omega\tilde{\vec{B}}_1$	(9.77)
$ik\hat{x}\times\tilde{\vec{B}}_1 = \mu_0 e n_0(\tilde{\vec{V}}_{i1} - \tilde{\vec{V}}_{e1}) - \frac{i\omega}{c^2}\tilde{\vec{E}}_1$	(9.78)

For un-magnetized plasma $\vec{B}_0 = 0$, and for $\vec{k} = \hat{x}k$, the governing equations in the (k,ω) domain can be obtained from Eqs. (9.24)~(9.33), i.e.,

If we wish to obtain the electrostatic wave, we can take the inner product $(i\hat{x}k/m_e)$ ·Eq. (9.71), which yields

$n_0(-i\omega)(ik)\tilde{V}_{e1x} = (k^2\tilde{p}_{e1} - en_0 ik\tilde{E}_{1x})/m_e$	(9.79)

Substituting Eq. (9.73) into Eq. (9.79) to eliminate $\tilde{p}_{e1}$, substituting Eq. (9.75) into Eq. (9.79) to eliminate $ik\tilde{E}_{1x}$, and then substituting Eq. (9.69) into the resulting equation to eliminate $ik\tilde{V}_{e1x}$, it yields

$(-i\omega)(i\omega)\tilde{n}_{e1} = k^2 \frac{\gamma_e p_{e0}}{m_e n_0} \tilde{n}_{e1} - \frac{e^2 n_0}{m_e \epsilon_0} (\tilde{n}_{i1} - \tilde{n}_{e1})$	(9.80)

Let $C_{e0} = \sqrt{\gamma_e p_{e0}/m_e n_0}$ and $\omega_{pe0} = \sqrt{e^2 n_0/m_e \epsilon_0}$, Eq. (9.80) can be rewritten as

$(\omega^2 - k^2 C_{e0}^2)\tilde{n}_{e1} + \omega_{pe0}^2(\tilde{n}_{i1} - \tilde{n}_{e1}) = 0$	(9.81)

Likewise, the inner product $(i\hat{x}k/m_i) \cdot$ Eq. (9.72) yields

$n_0(-i\omega)(ik)\tilde{V}_{i1x} = (k^2 \tilde{p}_{i1} + e n_0 ik \tilde{E}_{1x})/m_i$	(9.82)

Substituting Eq. (9.74) into Eq. (9.82) to eliminate $\tilde{p}_{i1}$, substituting Eq. (9.75) into Eq. (9.82) to eliminate $ik\tilde{E}_{1x}$, and then substituting Eq. (9.70) into the resulting equation to eliminate $ik\tilde{V}_{i1x}$, it yields

$$(-i\omega)(i\omega)\tilde{n}_{i1} = k^2 \frac{\gamma_i p_{i0}}{m_i n_0} \tilde{n}_{i1} + \frac{e^2 n_0}{m_e \epsilon_0} (\tilde{n}_{i1} - \tilde{n}_{e1}) \tag{9.83}$$

Let $C_{i0} = \sqrt{\gamma_i p_{i0}/m_i n_0}$ and $\omega_{pi0} = \sqrt{e^2 n_0/m_i \epsilon_0}$, Eq. (9.83) can be rewritten as

$$(\omega^2 - k^2 C_{i0}^2)\tilde{n}_{i1} - \omega_{pi0}^2(\tilde{n}_{i1} - \tilde{n}_{e1}) = 0 \tag{9.84}$$

Eq. (9.81) and Eq. (9.84) can be rewritten in the following matrix form

$$\begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \begin{bmatrix} \tilde{n}_{e1} \\ \tilde{n}_{i1} \end{bmatrix} = 0 \tag{9.85}$$

where

$$a = \frac{\omega_{pe0}^2}{\omega^2 - k^2 C_{e0}^2}$$

$$b = \frac{\omega_{pi0}^2}{\omega^2 - k^2 C_{i0}^2}$$

For $[\tilde{n}_{e1}, \tilde{n}_{i1}] \neq [0,0]$, it yields

$$\det\begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} = 0$$

i.e.,

$$(1-a)(1-b) - ab = 1 - a - b = 0$$

Thus, we obtain the linear dispersion relation of the electrostatic waves in the un-magnetized plasma

$$1 - \frac{\omega_{pe0}^2}{\omega^2 - k^2 C_{e0}^2} - \frac{\omega_{pi0}^2}{\omega^2 - k^2 C_{i0}^2} = 0 \qquad (9.86)$$

補充說明：

以後如果學雙流體（或多流體）不穩定 (two-stream instability or multi-stream instability)，當不同種類的流體以不同的平均速度 $\vec{V}_{\alpha 0}$ 運動時，將可得到類似 Eq. (9.86) 但不太相同的靜電波頻散關係式

$$1-\sum_{\alpha}\left[\frac{\omega_{p\alpha 0}^{2}}{(\omega-\vec{k}\cdot\vec{V}_{\alpha 0})^{2}-k^{2}C_{\alpha 0}^{2}}\right]=0 \qquad (9.87)$$

其中 $\omega-\vec{k}\cdot\vec{V}_{\alpha 0}$ 為 Doppler Shift 之後，該α種流體所感受到的靜電波頻率。

If we wish to obtain the electromagnetic wave, the cross product $(i\hat{x}k/m_e)\times$Eq. (9.71) yields

$$n_0(-i\omega)(i\hat{x}k)\times\tilde{\vec{V}}_{e1} = -en_0(i\hat{x}k/m_e)\times\tilde{\vec{E}}_1 \tag{9.88}$$

and the cross product $(i\hat{x}k/m_i)\times$Eq. (9.72) yields

$$n_0(-i\omega)(i\hat{x}k)\times\tilde{\vec{V}}_{i1} = en_0(i\hat{x}k/m_i)\times\tilde{\vec{E}}_1 \tag{9.89}$$

The cross product $(i\hat{x}k)\times$Eq. (9.78) yields

$$\begin{aligned}&(i\hat{x}k)\times(ik\hat{x}\times\tilde{\vec{B}}_1)\\ &\qquad = \mu_0 en_0(i\hat{x}k)\times(\tilde{\vec{V}}_{i1}-\tilde{\vec{V}}_{e1}) - \frac{i\omega}{c^2}(i\hat{x}k)\times\tilde{\vec{E}}_1\end{aligned} \tag{9.90}$$

Substituting Eq. (9.88) into Eq. (9.90) to eliminate $(i\hat{x}k)\times\tilde{\vec{V}}_{e1}$, and substituting Eq. (9.89) into Eq. (9.90) to eliminate $(i\hat{x}k)\times\tilde{\vec{V}}_{i1}$, it yields

$$(k^2\vec{\tilde{B}}_1) = \frac{i\omega}{c^2}(\frac{\omega_{pi0}^2 + \omega_{pe0}^2}{\omega^2} - 1)(i\hat{x}k)\times\vec{\tilde{E}}_1$$	(9.91)

Substituting Eq. (9.77) into Eq. (9.91) to eliminate $(i\hat{x}k)\times\vec{\tilde{E}}_1$, it yields

$$k^2\vec{\tilde{B}}_1 = \frac{1}{c^2}[\omega^2 - (\omega_{pi0}^2 + \omega_{pe0}^2)](\vec{\tilde{B}}_1)$$	(9.92)

For non-zero $\vec{\tilde{B}}_1$, Eq. (9.92), yields

$$\omega^2 = \left(\omega_{pi0}^2 + \omega_{pe0}^2\right) + c^2k^2$$	(9.93)

Since $\omega_{pi0}^2 \ll \omega_{pe0}^2$, Eq. (9.93) is very similar to the high-frequency electromagnetic wave dispersion relation obtained in Eq. (9.52). Likewise, Eq. (9.86) can be decomposed into two eigen modes, where the high-frequency one is very similar to the electrostatic wave dispersion relation obtained in Eq. (9.51).

以下簡單介紹，如何將 Eq. (9.86)拆成兩個 eigen modes.

Eq. (9.86) yields

	$(\omega^2 - k^2C_{e0}^2)\left(\omega^2 - k^2C_{i0}^2\right) - \omega_{pe0}^2\left(\omega^2 - k^2C_{i0}^2\right) - \omega_{pi0}^2(\omega^2 - k^2C_{e0}^2) = 0$
⇒	$(\omega^2)^2 - \left(k^2C_{e0}^2 + k^2C_{i0}^2 + \omega_{pe0}^2 + \omega_{pi0}^2\right)\omega^2 + \left(k^2C_{e0}^2k^2C_{i0}^2 + k^2C_{i0}^2\omega_{pe0}^2 + k^2C_{e0}^2\omega_{pi0}^2\right) = 0$
⇒	$(\omega^2)^2 - \left[(k^2C_{e0}^2 + \omega_{pe0}^2) + (k^2C_{i0}^2 + \omega_{pi0}^2)\right]\omega^2 + [(k^2C_{e0}^2 + \omega_{pe0}^2)(k^2C_{i0}^2 + \omega_{pi0}^2) - \omega_{pe0}^2\omega_{pi0}^2] = 0$
⇒	$(\omega^2)^2 - [a + b]\omega^2 + [ab - \omega_{pe0}^2\omega_{pi0}^2] = 0$ where $a = \left(k^2C_{e0}^2 + \omega_{pe0}^2\right) > 0$ $b = (k^2C_{i0}^2 + \omega_{pi0}^2) > 0$
⇒	$\omega^2 = \frac{1}{2}[(a + b) \pm \sqrt{(a + b)^2 - 4(ab - \omega_{pe0}^2\omega_{pi0}^2)}]$
⇒	For $T_{e0} \sim T_{i0}$, it yields $(b/a) \sim (m_e/m_i) \sim (1/1836)$, and
⇒	$\omega^2 \approx \frac{1}{2}[(a + b) \pm (a + b) \mp \frac{1}{2}\frac{4\left(ab - \omega_{pe0}^2\omega_{pi0}^2\right)}{(a + b)}]$

$\Rightarrow$	$$\omega^2 \approx \frac{1}{2}[(a+b) \pm (a+b) \mp 2(b - \frac{\omega_{pe0}^2 \omega_{pi0}^2}{a})]$$
	Since $$\frac{\omega_{pe0}^2 \omega_{pi0}^2}{a} = \frac{\omega_{pi0}^2}{1 + \frac{k^2 C_{e0}^2}{\omega_{pe0}^2}} = \frac{\omega_{pi0}^2}{1 + \gamma_e k^2 \lambda_{De0}^2}$$
$\Rightarrow$	$$\omega_H^2 \approx k^2 C_{e0}^2 + \omega_{pe0}^2 + k^2 C_{i0}^2 + \omega_{pi0}^2 - (k^2 C_{i0}^2 + \omega_{pi0}^2 - \frac{\omega_{pi0}^2}{1 + \gamma_e k^2 \lambda_{De0}^2})$$ $$\omega_L^2 \approx +(k^2 C_{i0}^2 + \omega_{pi0}^2 - \frac{\omega_{pi0}^2}{1 + \gamma_e k^2 \lambda_{De0}^2})$$
$\Rightarrow$	$$\omega_H^2 \approx \omega_{pe0}^2 + k^2 C_{e0}^2$$ $$\omega_L^2 \approx k^2 C_{i0}^2 + \omega_{pi0}^2 (1 - \frac{1}{1 + \gamma_e k^2 \lambda_{De0}^2})$$

The high-frequency electrostatic wave mode

$$\omega_H^2 = \omega_{pe0}^2 + k^2 C_{e0}^2$$	(9.94)

is the Langmuir wave obtained in Eq. (9.51).

The low-frequency (ion-time-scale) electrostatic wave mode is

$$\omega_L^2 = k^2 C_{i0}^2 + \omega_{pi0}^2 (1 - \frac{1}{1 + \gamma_e k^2 \lambda_{De0}^2}) \quad (9.95)$$

For $k^2 \lambda_{De0}^2 \ll 1$, we have

$$\frac{1}{1 + \gamma_e k^2 \lambda_{De0}^2} \approx 1 - \gamma_e k^2 \lambda_{De0}^2$$

Thus,

$$\omega_{pi0}^2 \left(1 - \frac{1}{1 + \gamma_e k^2 \lambda_{De0}^2}\right) \approx \omega_{pi0}^2 [1 - (1 - \gamma_e k^2 \lambda_{De0}^2)] = k^2 C_{e0}^2 \frac{\omega_{pi0}^2}{\omega_{pe0}^2}$$

It yields

$$\omega_L^2 = k^2 C_{i0}^2 + \omega_{pi0}^2 \left(1 - \frac{1}{1 + \gamma_e k^2 \lambda_{De0}^2}\right)$$

$$= k^2 C_{i0}^2 + k^2 C_{e0}^2 \frac{\omega_{pi0}^2}{\omega_{pe0}^2} = k^2 \frac{(\gamma_i k_B T_{i0} + \gamma_i k_B T_{e0})}{m_i}$$

Thus, for $k^2\lambda_{De0}^2 \ll 1$, Eq. (9.95) is reduced to

$$\omega_L^2 = k^2 C_{S0}^2 \qquad (9.96)$$

where $C_{S0} = \sqrt{(\gamma_i k_B T_{i0} + \gamma_i k_B T_{e0})/m_i}$ is the sound speed of the ion acoustic wave at the long wavelength limit in an un-magnetized plasma.

For $k^2\lambda_{De0}^2 \gg 1$, we have

$$\omega_{pi0}^2\left(1 - \frac{1}{1+\gamma_e k^2\lambda_{De0}^2}\right) \approx \omega_{pi0}^2\left[1 - \frac{1}{\gamma_e k^2\lambda_{De0}^2}\right] \approx \omega_{pi0}^2$$

Thus, for $k^2\lambda_{De0}^2 \gg 1$, Eq. (9.95) is reduced to

$$\omega_L^2 = k^2 C_{i0}^2 + \omega_{pi0}^2 \qquad (9.97)$$

This is the dispersion relation of the ion acoustic wave at the short wavelength limit in an un-magnetized plasma.

Based on the above discussion, the dispersion relations obtained in Eq. (9.93) and in Eqs. (9.94) & (9.95) are sketched in Figure 9.6.

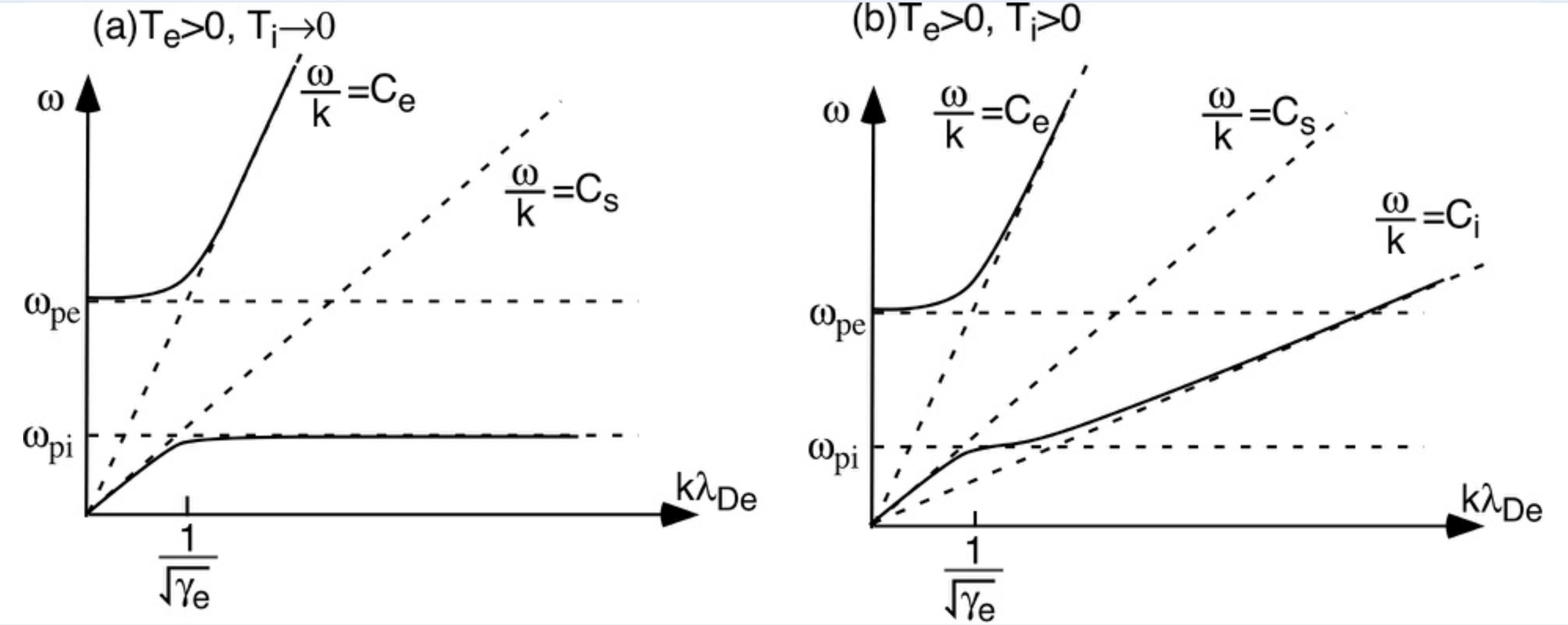

Figure 9.6. A sketch of the wave dispersion relation obtained a un-magnetized plasma based on Eqs. (9.93) ~ (9.95)

9.8. Ion-time-scale electrostatic waves in an un-magnetized plasma ($B_0 > 0$)

The dispersion relation obtained in Eq. (9.96) can be obtained by assuming that $n_e \approx n_i$, and by ignoring the electrons' inertial term in the electrons' momentum equation, i.e.,

$-i\omega\tilde{n}_{i1} = -n_0(ik)\tilde{V}_{i1x} \approx -i\omega\tilde{n}_{e1} = -n_0(ik)\tilde{V}_{e1x}$	(9.98)
$0 = -(ik)\tilde{p}_{e1} - en_0\tilde{E}_{1x}$	(9.99)
$m_i n_0(-i\omega)\tilde{V}_{i1x} = -(ik)\tilde{p}_{i1} + en_0\tilde{E}_{1x}$	(9.100)
$-i\omega\tilde{p}_{e1} = -\gamma_e p_{e0}(ik)\tilde{V}_{e1x}$	(9.101)
$-i\omega\tilde{p}_{i1} = -\gamma_i p_{i0}(ik)\tilde{V}_{i1x}$	(9.102)

Substituting Eq. (9.98) into Eqs. (9.101) & (9.102) to eliminate $(ik)\tilde{V}_{e1x}$ and $(ik)\tilde{V}_{i1x}$, it yields

$$\tilde{p}_{e1} = \gamma_e p_{e0} \tilde{n}_{i1}/n_0$$	(9.103)
$$\tilde{p}_{i1} = \gamma_i p_{i0} \tilde{n}_{i1}/n_0$$	(9.104)

Substituting Eq. (9.98) and (9.99) into (ik/m_i)Eq. (9.100) to eliminate $(ik)\tilde{V}_{i1x}$ and $en_0\tilde{E}_{1x}$ it yields

$$\omega^2 \tilde{n}_{i1} = (k^2 \tilde{p}_{i1} + k^2 \tilde{p}_{e1})/m_i$$	(9.105)

Substituting Eqs. (9.103) and (9.104) into Eq. (9.105) to eliminate $\tilde{p}_{i1}$ and $\tilde{p}_{e1}$, it yields

$$\omega^2 \tilde{n}_{i1} = k^2[(\gamma_i k_B T_{i0} + \gamma_e k_B T_{e0})/m_i]\tilde{n}_{i1}$$	(9.106)

Thus, for non-zero $\tilde{n}_{i1}$, Eq. (9.106) yields

$$\omega^2 = k^2 C_{S0}^2$$	(9.107)

where $C_{S0} = \sqrt{(\gamma_i k_B T_{i0} + \gamma_i k_B T_{e0})/m_i}$

Exercise 9.3:

Show that, although we have assumed that $n_e \approx n_i$, but the amplitude of the density perturbation of electrons should be slightly less than the density perturbation of ions ($|\tilde{n}_{e1}| \lesssim |\tilde{n}_{i1}|$) in the ion-acoustic wave.

9.9. Ion-time-scale waves in a magnetized plasma

As shown in Figure 9.7 that, for electron plasma frequency greater than electron's cyclotron frequency ($\omega_{pe0} > \Omega_{e0}$), the waves with frequency between the Ω_{e0} and Ω_{i0} consist of the whistler waves 哨波, the chorus waves 合唱波, and the ion-acoustic waves 正離子聲波. The ion-acoustic wave in the magnetized plasma is similar to the one discussed in section 9.8, except for nearly perpendicular propagation. The whistler waves and chorus waves are right-hand polarized electromagnetic waves or hybrid waves. The phase speed of the whistler wave increases with increasing wave frequency. The phase speed of the chorus wave decreases with increasing wave frequency.

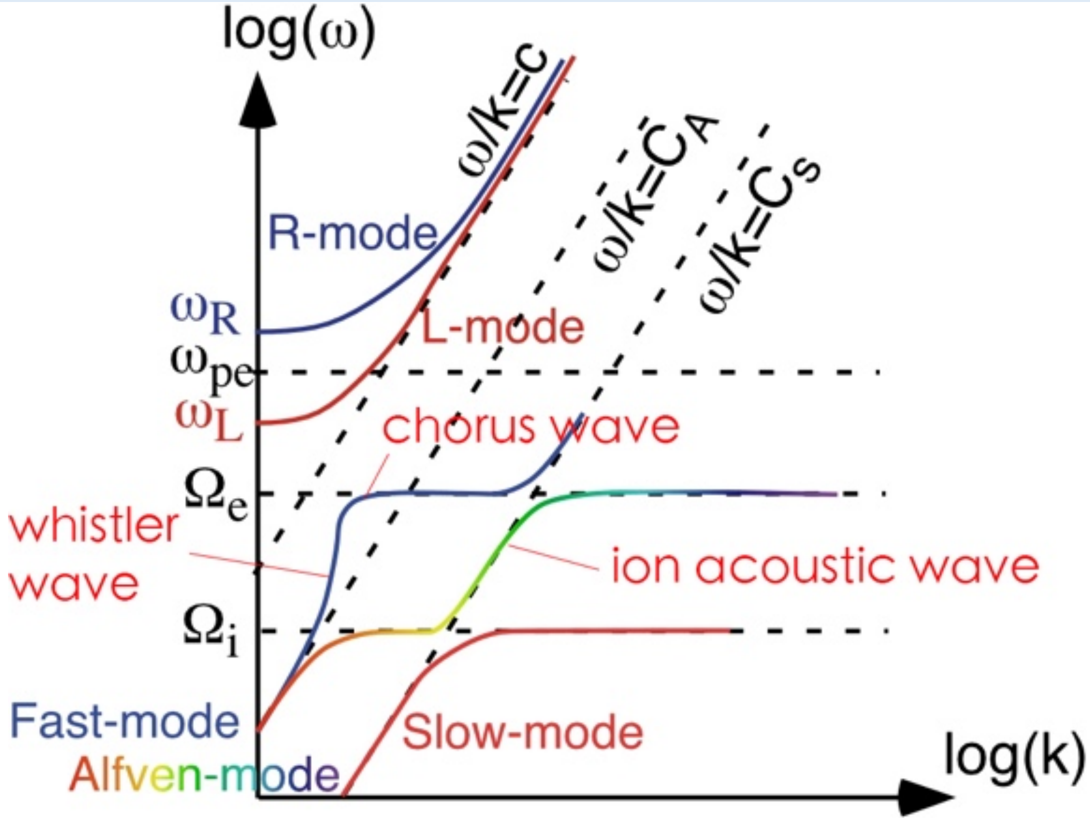

Figure 9.7. Dispersion relations of whistler waves 哨波, chorus waves 合唱波, and ion-acoustic waves 正離子聲波 in the magnetized ion-electron two-fluid plasma.