Lecture 8 Sound Wave in Neutral Gas

Key points

- Sound wave in neutral gas
- Longitudinal wave & transverse wave
- Linearize the governing equations
- Uniform and/or non-uniform equilibrium
- Plane wave assumption
- Fourier transform
- Dispersion relation
- Phase velocity and group velocity

8.1. Longitudinal Wave and Transverse Wave

- 電磁波是一種橫波。但是靜電波與聲波都屬於縱波。
- 橫波可以在有介質的環境中傳播，但是電磁波這種橫波也可以在真空中傳播。
- 縱波就一定要在介質中傳播。本講中我們將用中性大氣中的聲波為例，聲波的產生機制並推導線性的聲波方程式。
- 地震時，上下震動的 P 波是一種縱波，只有在靠近震央附近才能感受到它。左右搖晃的 S 波是一種橫波，可以傳到比較遠的地方。可見縱波 Damping 的速度比較快(why?)，所以離擾動源較遠的地方，通常只能測到橫波。

- 可是非線性的聲波，會以超音速的速度傳播，並且形成激震波(shock wave)，反而不易消失(why?)。
- The shock wave is an interested topic but is beyond the scope of this Lecture.

8.2. Basic Equations

Governing equations of the sound wave in a neutral gas are the fluid equations of the ideal gas. They are listed below

Continuity equation

$$\left(\frac{\partial}{\partial t} + \vec{V} \cdot \nabla\right)\rho = -\rho\nabla \cdot \vec{V} \tag{8.1}$$

Momentum equation

$$\rho\left(\frac{\partial}{\partial t} + \vec{V} \cdot \nabla\right)\vec{V} = -\nabla p + \rho\vec{g} \tag{8.2}$$

Adiabatic energy equation

$$\left(\frac{\partial}{\partial t} + \vec{V} \cdot \nabla\right)p = -\frac{5}{3}p\nabla \cdot \vec{V} \tag{8.3}$$

For sound wave $\nabla\cdot\vec{V}\neq 0$, therefore, Eq. (8.1) and (8.3) yields

$(\frac{\partial}{\partial t}+\vec{V}\cdot\nabla)p=\frac{5}{3}\frac{p}{\rho}(\frac{\partial}{\partial t}+\vec{V}\cdot\nabla)\rho$	(8.4)

$\left(\frac{\partial}{\partial t}+\vec{V}\cdot\nabla\right)$ Eq. (8.1) yields

$\left(\frac{\partial}{\partial t}+\vec{V}\cdot\nabla\right)^2\rho=-\left(\frac{\partial}{\partial t}+\vec{V}\cdot\nabla\right)\left(\rho\nabla\cdot\vec{V}\right)$ $=-\left(\nabla\cdot\vec{V}\right)\left(\frac{\partial}{\partial t}+\vec{V}\cdot\nabla\right)\rho-\rho\nabla\cdot\left(\frac{\partial}{\partial t}+\vec{V}\cdot\nabla\right)\vec{V}$ $=+\rho\left(\nabla\cdot\vec{V}\right)^2-\rho\nabla\cdot\left(-\frac{\nabla p}{\rho}+\vec{g}\right)$ $=+\rho(\nabla\cdot\vec{V})^2+\nabla^2 p-\frac{\nabla\rho\cdot\nabla p}{\rho}-\rho\nabla\cdot\vec{g}$	(8.5)

$\left(\frac{\partial}{\partial t}+\vec{V}\cdot\nabla\right)$ Eq. (8.4) yields

$$\left(\frac{\partial}{\partial t}+\vec{V}\cdot\nabla\right)^2\rho = \frac{1}{C_S^2}\left(\frac{\partial}{\partial t}+\vec{V}\cdot\nabla\right)^2 p + (1-\frac{5}{3})\rho(\nabla\cdot\vec{V})^2 \quad (8.6)$$

where $C_S^2 = (5/3)(p/\rho)$. Since, in the spatial scale of the sound wave, $\nabla\cdot\vec{g}\rightarrow 0$, we can ignore the $\nabla\cdot\vec{g}$ term in Eq. (8.5). Thus, Eqs. (8.5) and (8.6) yield

$$\nabla^2 p - \frac{1}{C_S^2}\left(\frac{\partial}{\partial t}+\vec{V}\cdot\nabla\right)^2 p = -\frac{5}{3}\rho(\nabla\cdot\vec{V})^2 + \frac{\nabla\rho\cdot\nabla p}{\rho} \quad (8.7)$$

Eq. (8.7) is the nonlinear sound wave equation under an assumption of that the compressional process is abiabatic.

For a uniform background equilibrium, each variable can be decomposed into an equilibrium component, denoted by a subscript "0" and a perturbed component, denoted by a subscript "1." That is $A(\vec{x},t) = A_0 + A_1(\vec{x},t)$. Thus, Eq. (8.7) becomes

$$\nabla^2 p_1 - \frac{1}{C_{S0}^2}[\frac{\partial}{\partial t} + (\vec{V}_0 + \vec{V}_1)\cdot\nabla]^2 p_1 = -\frac{5}{3}(\rho_0+\rho_1)(\nabla\cdot\vec{V}_1)^2 + \frac{\nabla\rho_1\cdot\nabla p_1}{(\rho_0+\rho_1)} \quad (8.8)$$

where $C_{S0}^2 = (5/3)(p_0/\rho_0)$. For small amplitude perturbation, $O(A_1/A_0) = O(\epsilon) < 10^{-3}$, we can ignore the second order, $O(\epsilon^2)$, or higher order terms in Eq. (8.8). It yields

$$\nabla^2 p_1 - \frac{1}{C_{S0}^2}\left(\frac{\partial}{\partial t} + \vec{V}_0\cdot\nabla\right)^2 p_1 = 0 \quad (8.9)$$

If we choose a moving frame in the background flow rest frame, it yields $\vec{V}_0 = 0$. Thus, Eq. (8.9) can be rewritten as

$$\nabla^2 p_1 - \frac{1}{C_{S0}^2}\frac{\partial^2 p_1}{\partial t^2} = 0 \qquad (8.10)$$

Eq. (8.10) is the linear sound wave equation. The sound wave propagates at a speed equal to the sound speed

$$C_{S0} = \sqrt{\frac{3}{5}\frac{p_0}{\rho_0}} = \sqrt{\frac{3}{5}\frac{k_B T_0}{m}} \qquad (8.11)$$

where $p_0 = n_0 k_B T_0$ and $\rho_0 = n_0 m$.

8.3. Linearizing the Fluid Equations

我們可以「先」線性化流體方程式，再來求聲波波動方程式。

如何線性化流體方程式呢？

Step 1: 先將所有變數拆解為平衡態 + 擾動態

$$A(\vec{x}, t) = A_0(\vec{x}) + A_1(\vec{x}, t) \quad (8.12)$$

Step 2: 寫出平衡態 + 擾動態的流體方程式。

Step 3: 寫出平衡態的流體方程式

Step 4: 將 Step 2 結果減去 Step 3 結果，得擾動態的方程式

Step 5: 若$O(A_1/A_0) = O(\epsilon) < 10^{-3}$，則消去等於或小於 $O(\epsilon^2)$, 的非線性項，就可得到線性化的擾動態方程式。

Substituting Eq. (8.12) into (8.1), the continuity equation becomes

$$[\frac{\partial}{\partial t} + (\vec{V}_0 + \vec{V}_1) \cdot \nabla](\rho_0 + \rho_1) = -(\rho_0 + \rho_1)\nabla \cdot (\vec{V}_0 + \vec{V}_1) \tag{8.13}$$

The equilibrium components of Eq. (8.1) satisfy the equilibrium continuity equation. i.e.,

$$\vec{V}_0 \cdot \nabla \rho_0 = -\rho_0 \nabla \cdot \vec{V}_0 \tag{8.14}$$

Subtracting Eq. (8.14) from Eq. (8.13) it yields the perturbed continuity equation

$$\left(\frac{\partial}{\partial t} + \vec{V}_0 \cdot \nabla\right)\rho_1 + \vec{V}_1 \cdot \nabla \rho_0 + \vec{V}_1 \cdot \nabla \rho_1 = -\rho_0 \nabla \cdot \vec{V}_1 - \rho_1 \nabla \cdot \vec{V}_0 - \rho_1 \nabla \cdot \vec{V}_1 \tag{8.15}$$

Substituting Eq. (8.12) into (8.2), the Eq. (8.2) becomes

$$(\rho_0+\rho_1)[\frac{\partial}{\partial t}+(\vec{V}_0+\vec{V}_1)\cdot\nabla](\vec{V}_0+\vec{V}_1) = -\nabla(p_0+p_1)+(\rho_0+\rho_1)\vec{g} \tag{8.16}$$

The equilibrium components of Eq. (8.2) satisfy the equilibrium momentum equation. i.e.,

$$\rho_0\vec{V}_0\cdot\nabla\vec{V}_0 = -\nabla p_0+\rho_0\vec{g} \tag{8.17}$$

Subtracting Eq. (8.17) from Eq. (8.16), it yields the perturbed momentum equation

$$\rho_0\left(\frac{\partial}{\partial t}+\vec{V}_0\cdot\nabla\right)\vec{V}_1+\rho_0\vec{V}_1\cdot\nabla\vec{V}_0+\rho_1\vec{V}_0\cdot\nabla\vec{V}_0 +\rho_0\vec{V}_1\cdot\nabla\vec{V}_1+\rho_1\vec{V}_1\cdot\nabla\vec{V}_0+\rho_1\left(\frac{\partial}{\partial t}+\vec{V}_0\cdot\nabla\right)\vec{V}_1 +\rho_1\vec{V}_1\cdot\nabla\vec{V}_1 = -\nabla p_1+\rho_1\vec{g} \tag{8.18}$$

Substituting Eq. (8.12) into (8.3), the Eq. (8.3) becomes

$$[\frac{\partial}{\partial t} + (\vec{V}_0 + \vec{V}_1) \cdot \nabla](p_0 + p_1) = -\frac{5}{3}(p_0 + p_1)\nabla \cdot (\vec{V}_0 + \vec{V}_1) \quad (8.19)$$

The equilibrium components of Eq. (8.3) satisfy the equilibrium adiabatic energy equation. i.e.,

$$\vec{V}_0 \cdot \nabla p_0 = -(5/3)p_0 \nabla \cdot \vec{V}_0 \quad (8.20)$$

Subtracting Eq. (8.20) from Eq. (8.19), it yields the perturbed adiabatic energy equation

$$\left(\frac{\partial}{\partial t} + \vec{V}_0 \cdot \nabla\right) p_1 + \vec{V}_1 \cdot \nabla p_0 + \vec{V}_1 \cdot \nabla p_1 = -\frac{5}{3} p_0 \nabla \cdot \vec{V}_1 - \frac{5}{3} p_1 \nabla \cdot \vec{V}_0 - \frac{5}{3} p_1 \nabla \cdot \vec{V}_1 \quad (8.21)$$

For small amplitude perturbation, $O(A_1/A_0) = O(\epsilon) < 10^{-3}$, we can ignore the second order, $O(\epsilon^2)$, or higher order terms in Eqs. (8.15), (8.18), & (8.21). The linearized fluid equations are :

The linearized continuity equation

$$\left(\frac{\partial}{\partial t} + \vec{V}_0 \cdot \nabla\right)\rho_1 + \vec{V}_1 \cdot \nabla \rho_0 = -\rho_0 \nabla \cdot \vec{V}_1 - \rho_1 \nabla \cdot \vec{V}_0 \quad (8.22)$$

The linearized momentum equation

$$\rho_0 \left(\frac{\partial}{\partial t} + \vec{V}_0 \cdot \nabla\right)\vec{V}_1 + \rho_0 \vec{V}_1 \cdot \nabla \vec{V}_0 + \rho_1 \vec{V}_0 \cdot \nabla \vec{V}_0 = -\nabla p_1 + \rho_1 \vec{g} \quad (8.23)$$

The linearized adiabatic energy equation

$$\left(\frac{\partial}{\partial t} + \vec{V}_0 \cdot \nabla\right)p_1 + \vec{V}_1 \cdot \nabla p_0 = -\frac{5}{3} p_0 \nabla \cdot \vec{V}_1 - \frac{5}{3} p_1 \nabla \cdot \vec{V}_0 \quad (8.24)$$

8.4. Sound Wave in a Medium With Uniform Background Flow

If the background flow velocity is uniform, all the spatial derivatives of the background flow velocity become zero. We can also choose the moving frame to be the equilibrium flow rest frame ($\vec{V}_0 = 0$). Thus, Eqs. (8.22)~(8.24) become

$\frac{\partial}{\partial t}\rho_1 = -\rho_0 \nabla \cdot \vec{V}_1 - \vec{V}_1 \cdot \nabla \rho_0$	(8.25)
$\rho_0 \frac{\partial}{\partial t}\vec{V}_1 = -\nabla p_1 + \rho_1 \vec{g}$	(8.26)
$\frac{\partial}{\partial t}p_1 = -\frac{5}{3}p_0 \nabla \cdot \vec{V}_1 - \vec{V}_1 \cdot \nabla p_0$	(8.27)

Likewise, the equilibrium momentum equation (8.17) becomes

$$-\nabla p_0 + \rho_0 \vec{g} = 0 \tag{8.28}$$

For $\vec{g} = -\hat{z}g$, Eq. (8.28) can be rewritten as

$$\frac{\partial p_0}{\partial x} = \frac{\partial p_0}{\partial y} = 0 \tag{8.29}$$

$$\frac{\partial p_0}{\partial z} = -\rho_0 g \tag{8.30}$$

Namely, $p_0 = p_0(z)$.

因為重力的關係，讓平衡態氣體壓力隨高度 z 的增加而遞減。

嚴格說來，波動如果沿著不均勻的平衡態方向傳播，波動的振幅、速度、與波長都會有很大的變化。也因此，沿著$\pm\hat{z}$ 方向傳播的波，也不會是簡單的正弦波或餘弦波。

8.5. Fourier Transform and Dispersion Relation

為了簡化起見，我們考慮沿著水平方向傳播的聲波。並取波前進方向為 $\hat{x}$ 方向。也就是 $\vec{k} = \hat{x}k$ 。我們做平面波的假設，其中波前所涵蓋的面，它的大小尺度遠大於波長，但是在z 方向的涵蓋面又不會太廣，所以可以忽略氣體壓力（甚至氣體密度）隨高度的變化情形。同樣的，我們也假設波的水平傳播距離，遠小於地球半徑，因此也可以忽略地球的曲率，以及地球旋轉所造成的科氏力。The above plane wave assumption yields

$$A_1(x,t) = Re\left\{\sum_n [\tilde{A}_1(k_n)e^{i[k_n x - \omega_n(k_n)t]}]\right\} \quad (8.28)$$

or simply

$A_1(x,t) = Re\{\tilde{A}_1(k)e^{i[kx-\omega(k)t]}\}$	(8.29)

where A_1 can be ρ_1, p_1, or V_{1x}. After Fourier transform, the PDE of $A_1(x,t)$ will be converted into an algebra equation of $\tilde{A}_1(k)$, where

$$\nabla = \hat{x}\frac{\partial}{\partial x} \rightarrow ik\hat{x} \text{ and } \frac{\partial}{\partial t} \rightarrow -i\omega$$

Namely, The Fourier transform of Eq. (8.25), the x-component of Eq.(8.26), and Eq. (8.27) becomes, respectively,

$-i\omega\tilde{\rho}_1 = -\rho_0(ik)\tilde{V}_{1x}$	(8.30)
$-i\omega\rho_0\tilde{V}_{1x} = -ik\tilde{p}_1$	(8.31)
$-i\omega\tilde{p}_1 = -(5/3)p_0(ik)\tilde{V}_{1x}$	(8.32)

Substituting Eq. (8.32) into ω(8.31) to eliminate $\tilde{p}_1$, it yields

$$\left(\frac{\omega^2}{k^2} - C_{S0}^2\right)\tilde{V}_{1x} = 0 \tag{8.33}$$

where $C_{S0}^2 = (5/3)(p_0/\rho_0)$.

我們也可以反過來將 Eq. (8.31) 代入ω(8.32) 消去 $\tilde{V}_{1x}$, 得到

$$\left(\frac{\omega^2}{k^2} - C_{S0}^2\right)\tilde{p}_1 = 0 \tag{8.34}$$

或將 Eq. (8.30) 代入 Eq. (8.33) 消去 $\tilde{V}_{1x}$, 得到

$$\left(\frac{\omega^2}{k^2} - C_{S0}^2\right)\frac{\omega}{k}\frac{\tilde{\rho}_1}{\rho_0} = 0 \tag{8.35}$$

For non-zero $\tilde{V}_{1x}$, $\tilde{p}_1$, and $\tilde{\rho}_1$, Eqs. (8.33)~(8.35) yields the "dispersion relation" of sound wave

$$\omega^2(k) = k^2 C_{S0}^2 \tag{8.36}$$

The phase velocity of the sound wave is

$$\vec{v}_{Ph} = \frac{\omega}{k}\hat{k} = \hat{k}C_{S0} \quad (8.37)$$

By definition, the group velocity is

$$\vec{v}_g = \nabla_{\vec{k}}\omega(\vec{k})$$

Thus, the group velocity of the sound wave is

$$\vec{v}_g = \hat{x}\frac{d\omega(k)}{dk} = \hat{x}C_{S0} = \vec{v}_{Ph} \quad (8.38)$$