Lecture 7 Introduction to Wave and Wave Equation

Key points

- Wave characteristics: wave front, phase, phase velocity, group velocity, longitudinal waves, and transverse waves
- Wave equation and the solutions of the wave equation
- Characteristic curves of the wave equation
- Electromagnetic waves travel through an empty space (a vacuum)
- Fourier transform and Fourier components of a linear wave: definition of linear wave, wavelength, wave number, wave period, wave frequency, wave angular frequency

7.1. Wave Characteristics （說明以下物理量的定義或意義）

- 波前 wave front
- 相位 phase
- 相速度 phase velocity
- 群速度 group velocity
- 縱波 longitudinal waves
- 橫波 transverse waves

如果有人問你，什麼是波動？用定性的文字來說明，非常冗長費事。比較簡單的說法是，滿足「波動方程式」的現象，就是波動。但是你知道，什麼是 wave equation 呢？

7.2. Wave Equations

Table 7.1 The 2nd-order Partial Differential Equations

二階偏微分方程式	對應之二次曲線方程式
2-D Poisson equation $$\frac{\partial^2 \Phi(x,y)}{\partial x^2} + \frac{\partial^2 \Phi(x,y)}{\partial y^2} = 1$$	elliptic equation $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
1-D diffusion equation $$\frac{\partial T(x,t)}{\partial t} = \kappa \frac{\partial^2 T(x,t)}{\partial x^2}$$	parabolic equation $$y = 4ax^2$$
1-D wave equation $$\frac{\partial^2 A(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 A(x,t)}{\partial t^2} = 0$$	hyperbolic equation $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

- Laplace equation $\nabla^2 \Phi\ (\vec{x}) = 0$ &
  Poisson equation $\nabla^2 \Phi\ (\vec{x}) = f(\vec{x})$ are elliptic equations.
- Diffusion equation is a parabolic equation.

$$\frac{\partial T(\vec{x},t)}{\partial t} = \kappa \nabla^2 T(\vec{x},t)$$

- Wave equation is a hyperbolic equation.

$$\nabla^2 A(\vec{x},t) - \frac{1}{c^2}\frac{\partial^2 A(\vec{x},t)}{\partial t^2} = S(\vec{x},t)$$

- 橢圓是一個封閉的曲線。因此，要解 Laplace equation or Poisson equation，需要提供函數的「邊界條件」。
- 拋物線是一個半封閉的曲線。因此，要解 diffusion equation 需要提供函數初始的空間分佈與邊界條件。

- 我們通常利用數值模擬 求 diffusion equation 與 wave equation 的解 。但是如果波速 c 是一個常數，我們還是可以得到 1-D wave equation 的解析解(analytical solutions) 。
- 雙曲線是一組開放的曲線。因此，要求 1-D wave equation 的解析解，我們需要提供函數兩組不同時間的空間分佈，或一組空間分佈與加一組邊界條件，或......。

下一節就簡單介紹，如何求得 1-D wave equation 的解析解。

7.3. Analytical Solutions of the One-Dimensional Wave Equation

Let us consider the following 1-D wave equation

$$\frac{\partial^2 A(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 A(x,t)}{\partial t^2} = 0 \qquad (7.1)$$

Since

$$\frac{\partial^2 A(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 A(x,t)}{\partial t^2} = \left[\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right]\left[\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right] A(x,t)$$

Eq. (7.1) yields

$$\left[\frac{\partial A(x,t)}{\partial x} + \frac{1}{c}\frac{\partial A(x,t)}{\partial t}\right] = 0 \qquad (7.2)$$

or

$$\left[\frac{\partial A(x,t)}{\partial x} - \frac{1}{c}\frac{\partial A(x,t)}{\partial t}\right] = 0 \qquad (7.3)$$

One can show that $A(x - ct)$ is the solution of Eq. (7.2).

Proof:

Substituting $A(x,t) = A(x - ct)$ into Eq. (7.2), it yields

$$\frac{\partial A(x-ct)}{\partial x} + \frac{1}{c}\frac{\partial A(x-ct)}{\partial t}$$

$$= \frac{dA(x-ct)}{d(x-ct)}\frac{\partial(x-ct)}{\partial x} + \frac{1}{c}\frac{dA(x-ct)}{d(x-ct)}\frac{\partial(x-ct)}{\partial t}$$

$$= \frac{dA(x-ct)}{d(x-ct)}\ 1 + \frac{1}{c}\frac{dA(x-ct)}{d(x-ct)}(-c) = 0$$

Likewise, $A(x+ct)$ is the solution of Eq. (7.3).
Solution of the wave equation (7.1) can be written as a linear combination of $F(x-ct)$ and $R(x+ct)$, that is,

$$A(x,t) = F(x-ct) + R(x+ct)$$

How do we know that $A(x-ct)$ is the solution of Eq. (7.2)?
本來 $A(x,t)$ 是兩個獨立變數 (x,t) 的函數，現在多了一個條件 Eq. (7.2) 就會讓獨立變數的數量減少一個。也就是說，我們可以找到一組新的獨立變數 $[\xi(x,t),\eta(x,t)]$ 使得 A 只是 ξ 的函數，而不是 η 的函數，也就是說

$$\left.\frac{\partial A}{\partial \eta}\right|_{\xi=\text{const.}} = 0 \qquad (7.4)$$

“若 $\xi(x,t)$ and $\eta(x,t)$ 的反函數存在”（這個條件很重要，因為有時不成立），則可將 x and t 寫成是 (ξ,η) 的函數，也就是 $x(\xi,\eta)$ and $t(\xi,\eta)$. 因此 Eq. (7.4) 可改寫為

$\left.\frac{\partial A[x(\xi,\eta),t(\xi,\eta)]}{\partial \eta}\right\|_{\xi} = \left.\frac{\partial A}{\partial x}\right\|_{t}\left.\frac{\partial x}{\partial \eta}\right\|_{\xi} + \left.\frac{\partial A}{\partial t}\right\|_{x}\left.\frac{\partial t}{\partial \eta}\right\|_{\xi} = 0$	(7.5)

比較 Eq. (7.5) 與 Eq. (7.2) 中 $\partial A/\partial x$ and $\partial A/\partial t$ 的係數

$$\left.\frac{\partial A}{\partial x}\right|_{t}\left.\frac{\partial x}{\partial \eta}\right|_{\xi} + \left.\frac{\partial A}{\partial t}\right|_{x}\left.\frac{\partial t}{\partial \eta}\right|_{\xi} = 0$$

$$\frac{\partial A}{\partial x} + \frac{1}{c}\frac{\partial A}{\partial t} = 0$$

可得

$$\left.\frac{\partial x}{\partial \eta}\right\|_{\xi} = 1$$	(7.6)
$$\left.\frac{\partial t}{\partial \eta}\right\|_{\xi} = \frac{1}{c}$$	(7.7)

Eq. (7.7) can be rewritten as

$$\left.\frac{\partial ct}{\partial \eta}\right\|_{\xi} = 1$$	(7.8)

Subtracting Eq. (7.8) from Eq. (7.6), it yields

$$\left.\frac{\partial (x-ct)}{\partial \eta}\right\|_{\xi} = 0$$	(7.9)

Eq. (7.9) yields, $x - ct$ is a function of ξ. The simplest solution is

$$x - ct = \xi(x, t)$$

Thus, the solution of Eq. (7.2) is $A(x,t) = A[\xi(x,t)] = A(x - ct)$.
Likewise, the solution of Eq. (7.3) is $A(x,t) = A(x + ct)$.

Table 7.2. Summary of 1-D wave equation and its solutions

一個二階偏微分方程式 $$\frac{\partial^2 A(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 A(x,t)}{\partial t^2} = 0$$ 可以降階為兩個一階偏微分方程式:	analytical solutions $A(x,t) = F(x - ct) + R(x + ct)$
$$\left[\frac{\partial A(x,t)}{\partial x} + \frac{1}{c}\frac{\partial A(x,t)}{\partial t}\right] = 0$$	analytical solution $A(x,t) = A(x - ct)$
$$\left[\frac{\partial A(x,t)}{\partial x} - \frac{1}{c}\frac{\partial A(x,t)}{\partial t}\right] = 0$$	analytical solution $A(x,t) = A(x + ct)$

7.4. Characteristic Curves of the 1-D Wave Equation

The characteristic curves of the equation (7.2) are $\xi(x,t) = x - ct = \text{constant curves}$. We can show that the phase of the perturbation is constant along a constant $\xi(x,t)$.

Figure 7.1 sketches the propagation of an initial disturbance based on Eq.(7.2). The amplitude of the disturbance is constant along each characteristic curve $\xi = x - ct$. The disturbance propagates toward $+x$ direction at a speed c.

Exercise 7.1. Describe the evolution of a disturbance which satisfies Eq. (7.1).

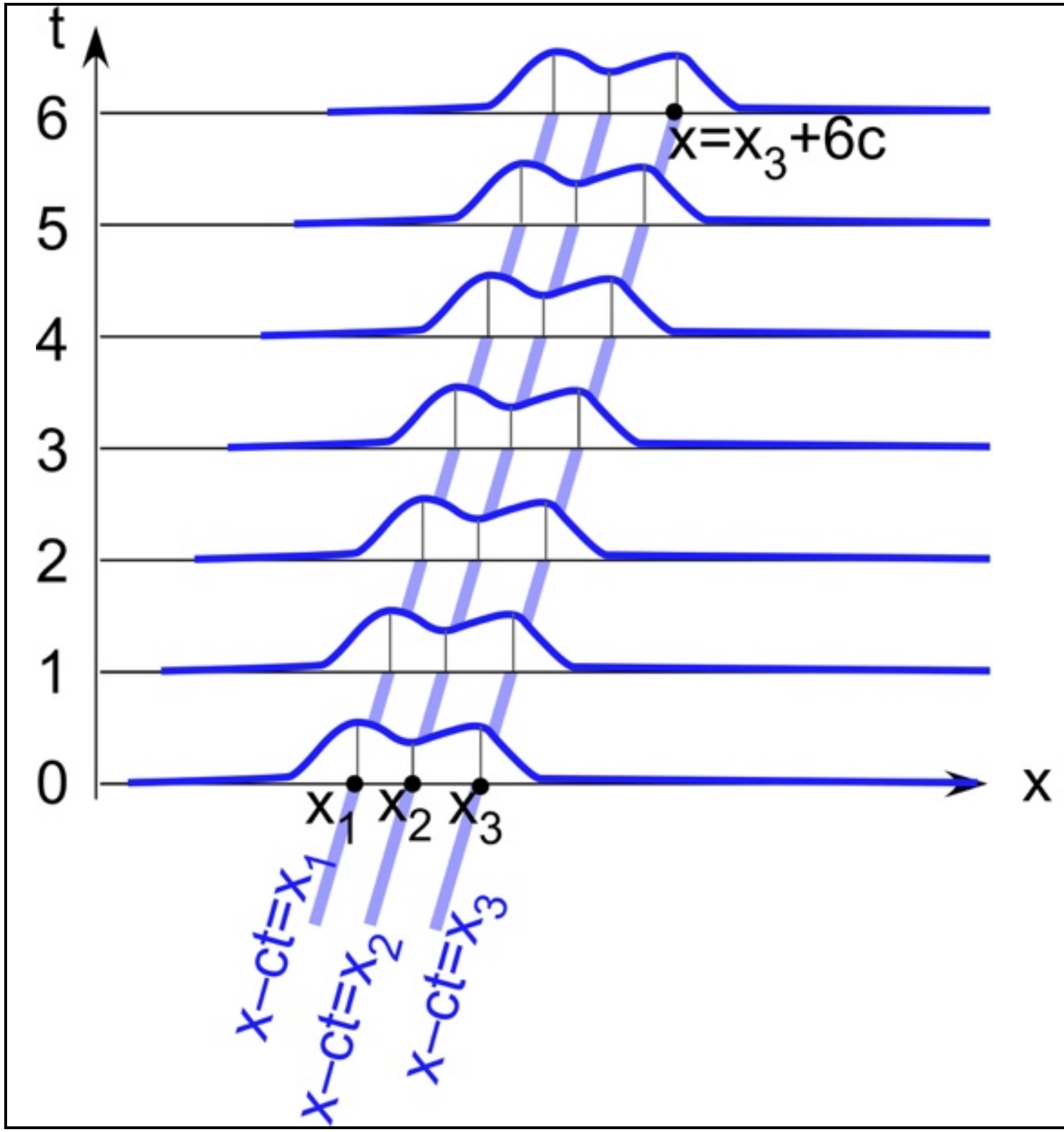

Figure 7.1. A sketch of the propagation of an initial disturbance based on Eq. (7.2). The amplitude of the disturbance is constant along each characteristic curve $\xi = x - ct$. The disturbance propagates toward $+x$ direction at a speed c.

7.5. Electromagnetic Wave Equation in a Vacuum

Let us consider a system without charge, electric current, or any dielectric medium. It yields $\vec{J} = 0$ and $\rho_c = 0$. The Maxwell equations in this system become

$\nabla \cdot \vec{E} = 0$	(7.10)
$\nabla \cdot \vec{B} = 0$	(7.11)
$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	(7.12)
$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$	(7.13)

The electromagnetic wave equation can be obtained by taking curl of the Faraday's Law (7.12) or the Ampere's Law (7.13).

$$\nabla\times\left(\nabla\times\vec{E}\right) = -\frac{\partial\nabla\times\vec{B}}{\partial t} = -\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla\times\left(\nabla\times\vec{B}\right) = \frac{1}{c^2}\frac{\partial\nabla\times\vec{E}}{\partial t} = -\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}$$

where

$$\nabla\times\left(\nabla\times\vec{E}\right) = -\nabla^2\vec{E} + \nabla\left(\nabla\cdot\vec{E}\right) = -\nabla^2\vec{E}$$

$$\nabla\times\left(\nabla\times\vec{B}\right) = -\nabla^2\vec{B} + \nabla(\nabla\cdot B) = -\nabla^2\vec{B}$$

Thus, we have

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0 \tag{7.14}$$

and

$$\nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0 \tag{7.15}$$

Eqs. (7.14) and (7.15) 就是真空中的電磁波方程式（electromagnetic wave equations in a vacuum）。

由 Eqs. (7.14) and (7.15) 可以看出，真空中的電磁波是以光速 c 在傳播。

補充說明一：這裡所謂的真空，其實是無介電質的環境。因此乾空氣對無線電波而言，可以視為真空。但是溼空氣，以及電離層電漿，對無線電波而言，就不能被視為是真空的環境。

補充說明二：如果環境中存在「介電質」則只有 curl of the Ampere's Law 可以得到純粹的電磁波方程式 (6.20)。若取 curl of the Faraday's Law 將會得到混合著電磁波與靜電波的方程式 (6.21)。

$\nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = -\mu_0\nabla\times\vec{J}$	(6.20)
$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \mu_0\frac{\partial\vec{J}}{\partial t} + \frac{1}{\epsilon_0}\nabla\rho_c$	(6.21)

因此，為了得到比較完整的電漿波動方程式，我們通常取 curl of the Faraday's Law，並用$\epsilon_0\nabla\left(\nabla\cdot\vec{E}\right)$ 取代(6.21)中的 $\nabla\rho_c$。

7.6. Fourier Transform & Fourier Components of a Linear Wave

少數的偏微分波動方程式可以利用 7.4 節的特徵曲線方法，找到解析解，大多數的偏微分方程式，需要仰賴數值模擬方式求解。另有一些波動現象，雖然原始的偏微分方程式非常複雜，但是對振幅比較小的波動，還是可以將原來的非線性波動方程進一步簡化成線性的波動方程式。如果我們要解的偏微分方程式已經被簡化成一個線性的偏微分方程式，我們就可以利用傅立葉轉換，將原來的線性偏微分方程式，轉換成一組代數方程式。這樣就可以很容易找出波動解的特性與形式。

Q: What is the definition of a linear function? and a linear wave?

Fourier Transform

For $0 < x < L$，任意函數 $f(x)$ 均可用 Fourier series 展開

$$f(x) = \sum_n [c_n \cos(k_n x) + d_n \sin(k_n x)]$$

此 Fourier series 的基底(basis)包括了偶函數 $\cos(k_n x)$ 與奇函數 $\sin(k_n x)$, 其中 $k_n = 2\pi n/L$ 。所以 c_n 就是 $f(x)$ 在 $\cos(k_n x)$ 上的投影， d_n 就是 $f(x)$ 在 $\sin(k_n x)$ 上的投影。

Let $c_n = r_n \cos(\phi_n)$ and $d_n = -r_n \sin(\phi_n)$. It yields

$$f(x) = \sum_n [r_n \cos(\phi_n) \cos(k_n x) - r_n \sin(\phi_n) \sin(k_n x)]$$

$$= \sum_n [r_n \cos(k_n x + \phi_n)]$$

Thus, the function $A(x-ct)$ can be written as

$$A(x-ct) = \sum_n \{\bar{A}(k_n)\cos[k_n(x-ct)+\phi_n]\}$$

$$= Re\left\{\sum_n [\bar{A}(k_n)e^{i\phi_n}e^{ik_n(x-ct)}]\right\}$$

where $\bar{A}(k_n)$ is the amplitude of the nth harmonic component in the Fourier series and ϕ_n is the phase of the nth harmonic component at $x=t=0$.

Let $\tilde{A}(k_n) = \bar{A}(k_n)e^{i\phi_n}$ and $k_n c = \omega_n$. It yields

$$A(x-ct) = Re\left\{\sum_n [\tilde{A}(k_n)e^{i(k_n x-\omega_n t)}]\right\} \qquad (7.16)$$

Since $A(x - ct)$ is a general solution of Eq. (7.2), substituting Eq. (7.16) into Eq. (7.2), it yields

$$\left[\frac{\partial}{\partial x} Re\left\{\sum_n [\tilde{A}(k_n) e^{i(k_n x - \omega_n t)}]\right\} + \frac{1}{c}\frac{\partial}{\partial t} Re\left\{\sum_n [\tilde{A}(k_n) e^{i(k_n x - \omega_n t)}]\right\}\right] = 0 \quad (7.17)$$

For

$$\frac{\partial}{\partial x} e^{i(k_n x - \omega_n t)} = i k_n e^{i(k_n x - \omega_n t)}$$

and

$$\frac{\partial}{\partial t} e^{i(k_n x - \omega_n t)} = -i\omega_n e^{i(k_n x - \omega_n t)}$$

Eq. (7.17) can be rewritten as

$$Re\left\{\sum_{n} \tilde{A}(k_n)(ik_n - \frac{i\omega_n}{c})\, e^{i(k_n x - \omega_n t)}\right\} = 0 \quad (7.18)$$

Eq. (7.18) yields

$$ik_n - \frac{i\omega_n}{c} = 0$$

or

$$\frac{\omega_n}{k_n} = c \quad (7.19)$$

Eq. (7.19) is the dispersion relation of the Eq. (7.2). Eq. (7.19) yields that the wave speed of the nth harmonic wave component is equal to ω_n/k_n, where $k_n = 2\pi n/L$ is the wave number, $\lambda_n = 2\pi/k_n = L/n$ is the wavelength.

Let the wave period of the nth harmonic wave component be τ_n, then, by definition, $c = \lambda_n / \tau_n$, or

$$\tau_n = \frac{\lambda_n}{c} = \frac{2\pi}{ck_n} = \frac{2\pi}{\omega_n} = \frac{1}{f_n}$$

Thus, ω_n is the angular frequency and f_n is the frequency of the nth harmonic wave component.

In summary, the so-called plane wave assumption,

$$A(\vec{x}, t) = Re\left\{\tilde{A}(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)}\right\}$$

which is a simplified form of Eq. (7.16), can turn a 3-D PDE into an algebra equation, where

$$\nabla = \frac{\partial}{\partial \vec{x}} \rightarrow i\vec{k} \text{ and } \frac{\partial}{\partial t} \rightarrow -i\omega$$

Q: What is plane wave? What is surface wave?

Exercise 7.2. Derive the wave dispersion relation of the EM waves in a vacuum.