Lecture 6

Component analysis of the $\boldsymbol{J}\times\boldsymbol{B}$ force, the electric field, and the current density field in plasma

Key points

- 忽略位移電流後， $\boldsymbol{J}\times\boldsymbol{B}$ force 可分解為磁壓梯度力與磁張力兩個分量
- 任何一個向量場(vector field) 都可分解成一個無旋的場分量（curl-free component）以及一個無散度的場分量（divergent-free component）
- 磁場是一個無散度的場，故磁場可寫為

$$\vec{B} = \nabla\times\vec{A}$$

- 電場可以分解為靜電場與電磁電場
  - 靜電場是一個無旋的電場分量
  - 電磁電場是一個無散度的電場分量

$$\vec{E} = \vec{E}^{E.S.} + \vec{E}^{E.M.} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

- 電流密度場也可拆解為一個無旋的靜電電流密度場，與一個無散度的電磁電流密度場。
- 磁場與電場之波動方程式

6.1. 分解 $\boldsymbol{J}\times\boldsymbol{B}$ force

If we ignore the displacement current ($\epsilon_0\mu_0\partial\vec{E}/\partial t \to 0$), the $\boldsymbol{J}\times\boldsymbol{B}$ Lorentz force can be decomposed into a magnetic pressure gradient force and a magnetic tension force. That is

$$\vec{J}\times\vec{B} \approx \frac{\nabla\times\vec{B}}{\mu_0}\times B = \frac{\vec{B}\cdot\nabla\vec{B}}{\mu_0} - \frac{\nabla B^2}{2\mu_0}$$
$$= -\frac{\nabla_\perp B^2}{2\mu_0} - \frac{\nabla_\parallel B^2}{2\mu_0} + \frac{\vec{B}\cdot\nabla\vec{B}}{\mu_0} \qquad (6.1)$$

where $\nabla_\parallel = \hat{B}\hat{B}\cdot\nabla$ and $\nabla_\perp = \nabla - \nabla_\parallel = (\overset{\rightrightarrows}{1} - \hat{B}\hat{B})\cdot\nabla$. Since

$$\frac{\hat{R}_B}{R_B} = -\hat{B}\cdot\nabla\hat{B} = -\frac{\vec{B}\cdot\nabla\vec{B}}{B^2} + \frac{\nabla_\parallel B}{B} \qquad (6.2)$$

Substituting Eq. (6.2) into Eq. (6.1), it yields

$$\vec{J}\times\vec{B} = -\nabla_{\perp}\left(\frac{B^2}{2\mu_0}\right) - \frac{\hat{R}_B}{R_B}\frac{B^2}{\mu_0}$$	(6.3)
magnetic pressure gradient force; magnetic tension force	

6.2. 向量恆等式及其應用

以下幾個向量恆等式，非常好用。透過對 Maxwell's equations 的認識，也可幫助我們記憶這些向量恆等式。

$\nabla \times \nabla f = 0$	(6.4)
$\nabla \cdot (\nabla \times \vec{A}) = 0$	(6.5)
$\nabla \cdot (\nabla f \times \nabla g) = 0$	(6.6)
$\hat{s} \cdot \nabla f = \frac{df}{ds}$	(6.7)

Application (1) : Initial conditions of $\vec{E}$ and $\vec{B}$

Equation (6.5) yields $\nabla \cdot \left(\nabla \times \vec{B}\right) = 0$ and $\nabla \cdot \left(\nabla \times \vec{E}\right) = 0$
Thus, taking divergence of the following equations

$$\frac{\partial \vec{E}}{\partial t} = c^2 \nabla \times \vec{B} - \frac{1}{\epsilon_0}\vec{J}$$	(6.8)
$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$$	(6.9)

it yields

$$\frac{\partial (\nabla \cdot \vec{E})}{\partial t} = -\frac{1}{\epsilon_0}\nabla \cdot \vec{J} = \frac{1}{\epsilon_0}\frac{\partial \rho_c}{\partial t}$$	(6.10)

$$\frac{\partial(\nabla \cdot \vec{B})}{\partial t} = 0 \qquad (6.11)$$

where the charge continuity has been used to eliminate $\nabla \cdot \vec{J}$ in Eq. (6.10). Equation (6.10) yields that the initial condition of the Electric field in the Ampere's Law satisfies

$$\nabla \cdot \vec{E} = \frac{\rho_c}{\epsilon_0}$$

Equation (6.11) yields that the initial condition of the magnetic field in the Faraday's Law satisfies

$$\nabla \cdot \vec{B} = 0$$

Application (2) : Introduction to vector potential

$$\nabla \cdot \vec{B} = 0$$

Eq. (6.5) yields that the magnetic field can be expressed in the following form

$$\vec{B} = \nabla \times \vec{A} \tag{6.12}$$

where $\vec{A}$ is the vector potential.

Application (3) : Introduction to scalar potential

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$$

For electrostatic (E.S.) wave of phenomena $\partial \vec{B}/\partial t = 0$, it yields

$\nabla \times \vec{E}^{E.S.} = 0$	(6.13)

Eq. (6.4) yields that the electrostatic electric field ($\vec{E}^{E.S.}$) can be casted in the following form

$\vec{E}^{E.S.} = -\nabla \phi$	(6.14)

The Poisson Equation yields

$$\nabla \cdot \vec{E}^{E.S.} = -\nabla^2 \phi = \frac{\rho_c}{\epsilon_0}$$

Application (4) : Decomposing the electric field $\vec{E}$

The electric field can be decomposed into a curl-free electrostatic component ($\vec{E}^{E.S.}$) and a divergence-free electromagnetic component ($\vec{E}^{E.M.}$). We have shown that the curl-free electrostatic electric field satisfies

$$\nabla \times \vec{E}^{E.S.} = 0$$

and can be casted into

$$\vec{E}^{E.S.} = -\nabla \phi$$

The divergence-free electromagnetic (E.M.) electric field ($\vec{E}^{E.M.}$) satisfies

$$\nabla \cdot \vec{E}^{E.M.} = 0 \qquad (6.15)$$

and

$\nabla\times\vec{E}^{E.M.}=-\frac{\partial\vec{B}}{\partial t}$	(6.16)

Substituting Eq. (6.12) $\vec{B}=\nabla\times\vec{A}$ into the above Eq. (6.16) Faraday's Law, it yields,

$\vec{E}^{E.M.}=-\frac{\partial\vec{A}}{\partial t}$	(6.17)

By definition, $\nabla\cdot\vec{E}^{E.M.}=0$, it yields $\nabla\cdot\vec{A}=0$, which is called the Coulomb gauge. As a result, the total electric field can be written in two components

$\vec{E}=\vec{E}^{E.S.}+\vec{E}^{E.M.}=-\nabla\phi-\frac{\partial\vec{A}}{\partial t}$	(6.18)

Application (5) : Decomposing the current density field $\vec{J}$

The current density field can be decomposed into a curl-free electrostatic current density component ($\vec{J}^{E.S.}$) and a divergence-free electromagnetic current density component ($\vec{J}^{E.M.}$). It yields the current density in the charge continuity is the electrostatic current density. Namely,

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot \vec{J}^{E.S.} = 0 \qquad (6.19)$$

The current density in the Ampere's Law consists of both electrostatic and electromagnetic components. Taking a curl of the Ampere's Law, we can obtain an equation of the electromagnetic current density. That is

$$\nabla\times(\nabla\times\vec{B}) = \mu_0\nabla\times\vec{J} + \frac{1}{c^2}\frac{\partial\nabla\times\vec{E}}{\partial t}$$

It yields only the divergence-free components of the vector fields. That is

$$-\nabla^2\vec{B} = \mu_0\nabla\times\vec{J}^{E.M.} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}$$

or

$$\nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = -\mu_0\nabla\times\vec{J}^{E.M.} = -\mu_0\nabla\times\vec{J} \qquad (6.20)$$

Eq. (6.20) is the wave equation of the magnetic field with a source term that is proportional to $\nabla\times\vec{J}$.

Likewise, taking a curl of the Faraday's Law, it yields

$$\nabla\times\left(\nabla\times\vec{E}\right) = -\frac{\partial\nabla\times\vec{B}}{\partial t} = -\mu_0\frac{\partial\vec{J}}{\partial t} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$$

For $\nabla\times\left(\nabla\times\vec{E}\right) = -\nabla^2\vec{E} + \nabla(\nabla\cdot\vec{E})$, we have

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \mu_0\frac{\partial\vec{J}}{\partial t} + \frac{1}{\epsilon_0}\nabla\rho_c \qquad (6.21)$$

Eq. (6.21) is the wave equation of the electric field with two source terms. One of them is proportional to $\partial\vec{J}/\partial t$. The other one is proportional to $\nabla\rho_c$. Eq. (6.21) consists of both the electrostatic component and the electromagnetic component for the fields $\vec{E}$ and $\vec{J}$.

6.3. 理論與觀測上的應用

第 **6.2** 節所獲得的結果，「目前」在觀測上的應用不多，因為「目前」太空觀測的空間解析度太差，無法計算各向量場的旋度與散度。可是在理論與模擬的資料分析上，卻可以提供很有用的資訊。透過分析各向量場的旋度與散度，可以很容易的展現二維或三維的向量場資料，並提供重要的物理訊息與因果關係。在不久的將來，觀測技術或許可以大幅提升空間解析度，使科學家有能力從觀測資料計算各向量場的旋度與散度時。屆時，第 **6.2** 節所討論的內容，也將有助於觀測資料的分析。