# Appendix A General Solutions of the Poisson Equation

Green's function of the Poisson equation

$$\nabla^2 G(\vec{r}) = \delta(\vec{r}) \qquad \text{(A.1)}$$

Where $\delta(\vec{r})$ is a delta function in a 3-dimensional system.

Let $\delta(r)$ be a 1-dimensional delta function. since

$$1 = \iiint \delta(\vec{r}) d^3x = \iiint \frac{\delta(r)}{4\pi r^2} r^2 \sin\theta \; dr \, d\theta \, d\phi = \int \delta(r) dr$$

It yields

$$\delta(\vec{r}) = \frac{\delta(r)}{4\pi r^2} \qquad \text{(A.2)}$$

Substituting Equation (A.2) into Equation (A.1), and consider isotropic Green's function $G(r)$, it yields

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\,G(r)\right]=\frac{\delta(r)}{4\pi r^2} \tag{A.3}$$

Multiplying Equation (A.3) by $r^2$, it yields

$$\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\,G(r)\right]=\frac{\delta(r)}{4\pi} \tag{A.4}$$

Integrating Equation (A.4) once w.r.t. $r$, it yields

$$\left[r^2\frac{\partial}{\partial r}\,G(r)\right]=\frac{1}{4\pi} \tag{A.5}$$

Multiplying Equation (A.5) by $1/r^2$, it yields

$$\frac{\partial}{\partial r}\,G(r)=\frac{1}{4\pi r^2} \tag{A.6}$$

Integrating Equation (A.6) once w.r.t. $r$, it yields

$$G(r) = -\frac{1}{4\pi r} \tag{A.7}$$

Now, let us consider the gravitational potential equation

$$\nabla^2 \Phi_g(\vec{r}) = 4\pi\rho(\vec{r}) \tag{A.8}$$

where $\vec{g} = -\nabla\Phi_g(\vec{r})$. Since Poisson Equation is a linear differential equation and since the mass density cab be written as

$$\rho(\vec{r}) = \iiint_{Vol.} \delta(\vec{r} - \vec{r}\,')\rho(\vec{r}\,')d\vec{r}\,' \tag{A.9}$$

The solution of Equation (A.8) can be obtained from Equations (A.1), (A.7), and (A.9). Namely, we have

$$\Phi_g(\vec{r}) = 4\pi \iiint_{Vol.} G(\vec{r} - \vec{r}')\rho(\vec{r}')d\vec{r}' \qquad \text{(A.10)}$$

Substituting (A.7) into (A.10), it yields

$$\Phi_g(\vec{r}) = 4\pi \iiint_{Vol.} -\frac{\rho(\vec{r}')}{4\pi|\vec{r} - \vec{r}'|} d\vec{r}'$$

or

$$\Phi_g(\vec{r}) = -\iiint_{Vol.} \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d\vec{r}' \qquad \text{(A.11)}$$

The gravitational field can be obtained from $\vec{g} = -\nabla \Phi_g(\vec{r})$

$$\vec{g}(\vec{r}) = -\iiint_{Vol.} \frac{\rho(\vec{r}'\,)}{\left|\vec{r} - \vec{r}'\,\right|^2} \frac{(\vec{r} - \vec{r}'\,)}{\left|\vec{r} - \vec{r}'\,\right|} \; d\vec{r}' \qquad \text{(A.12)}$$

Likewise, the electrostatic electric field, which satisfies $\nabla \times \vec{E}^{ES} = 0$ and the Gauss law for electric field

$$\nabla \cdot \vec{E}^{ES} = \frac{\rho_c}{\epsilon_0}$$

The curl-free condition yields $\vec{E}^{ES}$ can be written as $\vec{E}^{ES} = -\nabla \Phi$. Thus, the electric potential satisfies the following Poisson equation

$$\nabla^2 \Phi(\vec{r}) = -\frac{\rho_c(\vec{r})}{\epsilon_0} \qquad \text{(A.13)}$$

The solution of the electric potential can be written as

$$\Phi(\vec{r}) = \frac{1}{\epsilon_0} \iiint_{Vol.} \frac{\rho_c(\vec{r}')}{4\pi|\vec{r} - \vec{r}'|} d\vec{r}' \qquad \text{(A.14)}$$

The electrostatic electric field can be obtained from $\vec{E}^{ES} = -\nabla\Phi$, it yields

$$\vec{E}^{ES}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iiint_{Vol.} \frac{\rho_c(\vec{r}')}{|\vec{r} - \vec{r}'|^2} \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|} d\vec{r}' \qquad \text{(A.15)}$$

Let us consider a time-independent magnetic field $\vec{B}(\vec{r})$ (靜磁場). Since $\vec{B}(\vec{r})$ satisfies $\nabla \cdot \vec{B} = 0$, we can define a vector potential $\vec{A}(\vec{r})$, such that $\vec{B} = \nabla \times \vec{A}(\vec{r})$. The time-independent Ampere's Law can be written as

$$\nabla \times \left( \nabla \times \vec{A}(\vec{r}) \right) = \mu_0 \vec{J} \tag{A.16}$$

If we choose the Coulomb gauge $\nabla \cdot \vec{A}(\vec{r}) = 0$, Equation (A.16) can be rewritten in the following form (three-components Poisson Equations)

$$\nabla^2 \vec{A}(\vec{r}) = -\mu_0 \vec{J} \tag{A.17}$$

The solution of Equation (A.17) can be written as

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint_{Vol.} \frac{\vec{J}(\vec{r}'\,)}{\left|\vec{r} - \vec{r}'\,\right|} d\vec{r}' \qquad \text{(A.18)}$$

The static magnetic field can be obtained from $\vec{B} = \nabla \times \vec{A}$, it yields

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint_{Vol.} \frac{\vec{J}(\vec{r}'\,)}{\left|\vec{r} - \vec{r}'\,\right|^2} \times \frac{(\vec{r} - \vec{r}'\,)}{\left|\vec{r} - \vec{r}'\,\right|} d\vec{r}' \qquad \text{(A.19)}$$