Lecture 4. Frozen-in Flux in Magnetohydrodynamic Plasma

The so-called ideal magnetohydrodynamic (MHD) plasma model is a model designed for studying low-frequency long-wavelength plasma phenomena.

Ohm's Law in MHD limit (i.e., in low-frequency, long-wavelength limit) can be written as $\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$.

Exercise 4.1.

(a) Show that electrostatic potential is constant along streamline and magnetic filed line *in steady-state MHD plasma*. (i.e., Constant potential surface is determined by a set of streamlines and magnetic field lines *in steady-state MHD plasma*.)

(b) What will happen if there is a potential difference along magnetic field line in MHD plasma?

(c) What will happen if there is a potential difference along streamline in MHD plasma?

In this lecture, we shall use two different approaches to show that if a plasma fluid satisfies $\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$ then the magnetic flux is frozen in the plasma, i.e.,

$$\boxed{\frac{d\Phi_B}{dt} = \frac{d}{dt} \int\int_{S(t)} \mathbf{B}(\mathbf{x},t) \cdot d\mathbf{a} = 0} \qquad (4.1)$$

where d/dt is a physical notation (but not a mathematical notation) of time derivatives along the path of a fluid element.

The following three equations are the sufficient conditions of Eq.(4.1).

$\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$	(MHD Ohm's Law, or MHD approximation)	(4.2)
$\nabla \cdot \mathbf{B} = 0$	(No magnetic monopole)	(4.3)
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	(Faraday's Law)	(4.4)

4.1. Proof of Frozen-in Flux (Method 1)

By definition, variation of magnetic flux along the path line of fluid elements is

$$\frac{d\Phi_B}{dt} = \frac{d}{dt} \int\int_{S(t)} \mathbf{B} \cdot d\mathbf{a} = \lim_{\Delta t \to 0} \frac{\int\int_{S(t+\Delta t)} \mathbf{B}(\mathbf{x}, t+\Delta t) \cdot d\mathbf{a} - \int\int_{S(t)} \mathbf{B}(\mathbf{x},t) \cdot d\mathbf{a}}{\Delta t} \qquad (4.5)$$

Since $\nabla \cdot \mathbf{B} = 0$, it yields

$$0 = \iiint \nabla \cdot \mathbf{B} d^3x = \oint\!\!\oint \mathbf{B}(\mathbf{x}, t+\Delta t) \cdot d\mathbf{a}$$

$$= \int\int_{S(t+\Delta t)} \mathbf{B}(\mathbf{x}, t+\Delta t) \cdot d\mathbf{a} - \int\int_{S(t)} \mathbf{B}(\mathbf{x}, t+\Delta t) \cdot d\mathbf{a} + \oint \mathbf{B}(\mathbf{x}, t+\Delta t) \cdot (d\mathbf{l} \times \mathbf{V}\Delta t)$$

$$= \int\int_{S(t+\Delta t)} \mathbf{B}(\mathbf{x}, t+\Delta t) \cdot d\mathbf{a} - \int\int_{S(t)} \mathbf{B}(\mathbf{x}, t+\Delta t) \cdot d\mathbf{a} + \oint d\mathbf{l} \cdot [\mathbf{V}\Delta t \times \mathbf{B}(\mathbf{x}, t+\Delta t)]$$

or

$$\int\int_{S(t+\Delta t)} \mathbf{B}(\mathbf{x}, t+\Delta t) \cdot d\mathbf{a} = \int\int_{S(t)} \mathbf{B}(\mathbf{x}, t+\Delta t) \cdot d\mathbf{a} - \oint d\mathbf{l} \cdot [\mathbf{V}\Delta t \times \mathbf{B}(\mathbf{x}, t+\Delta t)] \qquad (4.6)$$

where the corresponding surface integrations are sketched in Figure 4.1.

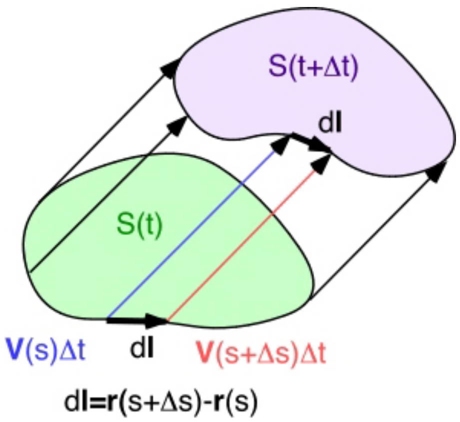

Figure 4.1. Sketches of surface integration domain discussed in Eq. (4.6).

Substituting Eq. (4.6) into Eq. (4.5) yields

$$\begin{aligned}\frac{d\Phi_B}{dt} &= \lim_{\Delta t\to 0}\frac{\int\int_{S(t+\Delta t)}\mathbf{B}(\mathbf{x},t+\Delta t)\cdot d\mathbf{a} - \int\int_{S(t)}\mathbf{B}(\mathbf{x},t)\cdot d\mathbf{a}}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{[\int\int_{S(t)}\mathbf{B}(\mathbf{x},t+\Delta t)\cdot d\mathbf{a} - \oint d\mathbf{l}\cdot[\mathbf{V}\Delta t\times\mathbf{B}(\mathbf{x},t+\Delta t)]] - \int\int_{S(t)}\mathbf{B}(\mathbf{x},t)\cdot d\mathbf{a}}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{\int\int_{S(t)}[\mathbf{B}(\mathbf{x},t+\Delta t)-\mathbf{B}(\mathbf{x},t)]\cdot d\mathbf{a}}{\Delta t} - \oint d\mathbf{l}\cdot(\mathbf{V}\times\mathbf{B})\\ &= \int\int_S\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{a} - \oint d\mathbf{l}\cdot(\mathbf{V}\times\mathbf{B})\end{aligned} \tag{4.7}$$

Substituting Eq. (4.4) into Eq. (4.7) and then substituting Eq. (4.2) into the resulting equation, it yields

$$\begin{aligned}\frac{d\Phi_B}{dt} &= \int\int_S\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{a} - \oint d\mathbf{l}\cdot(\mathbf{V}\times\mathbf{B})\\ &= \int\int_S(-\nabla\times\mathbf{E})\cdot d\mathbf{a} - \oint d\mathbf{l}\cdot(\mathbf{V}\times\mathbf{B})\\ &= \oint d\mathbf{l}\cdot(-\mathbf{E}) - \oint d\mathbf{l}\cdot(\mathbf{V}\times\mathbf{B})\\ &= \oint d\mathbf{l}\cdot(\mathbf{V}\times\mathbf{B}) - \oint d\mathbf{l}\cdot(\mathbf{V}\times\mathbf{B})\\ &= 0\end{aligned} \tag{4.8}$$

Since, Eq. (4.8) is consistent with Eq. (4.1), we have successfully proved that the magnetic flux is frozen in MHD plasma.

4.2. Proof of Frozen-in Flux (Method 2)

Since $\nabla\cdot\mathbf{B}=0$, we can let

$$\mathbf{B}=\nabla\times\mathbf{A} \tag{4.9}$$

Substituting Eq. (4.9) into Eq. (4.4) becomes

$$\mathbf{E}^{EM} = -\frac{\partial\mathbf{A}}{\partial t} \tag{4.10}$$

Since $\mathbf{E}^{ES} = -\nabla\Phi$, the total electric field can be written as

$$\mathbf{E}=\mathbf{E}^{EM}+\mathbf{E}^{ES}=-(\partial\mathbf{A}/\partial t)-\nabla\Phi$$

or

$$\partial \mathbf{A} / \partial t = -\mathbf{E} - \nabla \Phi \tag{4.11}$$

Substituting Eq. (4.9) into Eq. (4.1) yields

$$\begin{aligned}
\frac{d\Phi_B}{dt} &= \frac{d}{dt} \int \int_{S(C)} (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \frac{d}{dt} \oint_C \mathbf{A} \cdot d\mathbf{l} = \oint_C \frac{d\mathbf{A}}{dt} \cdot d\mathbf{l} + \oint_C \mathbf{A} \cdot \frac{d}{dt}(d\mathbf{l}) \\
&= \oint_C (\frac{\partial \mathbf{A}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{A}) \cdot d\mathbf{l} + \oint_C \mathbf{A} \cdot \frac{d}{dt}[\mathbf{r}(s+\Delta s, t) - \mathbf{r}(s,t)] \\
&= \oint_C (\frac{\partial \mathbf{A}}{\partial t}) \cdot d\mathbf{l} + \oint_C (\mathbf{V} \cdot \nabla \mathbf{A}) \cdot d\mathbf{l} + \oint_C \mathbf{A} \cdot [\mathbf{V}(s+\Delta s, t) - \mathbf{V}(s,t)]
\end{aligned} \tag{4.12}$$

Substituting Eq. (4.11) into Eq. (4.12), and then substituting Eqs. (4.2) and (4.9) into the resulting equation, it yields

$$\begin{aligned}
\frac{d\Phi_B}{dt} &= \oint_C (\frac{\partial \mathbf{A}}{\partial t}) \cdot d\mathbf{l} + \oint_C (\mathbf{V} \cdot \nabla \mathbf{A}) \cdot d\mathbf{l} + \oint_C \mathbf{A} \cdot [\mathbf{V}(s+\Delta s, t) - \mathbf{V}(s,t)] \\
&= \oint_C (-\mathbf{E} - \nabla \Phi) \cdot d\mathbf{l} + \oint_C (\mathbf{V} \cdot \nabla \mathbf{A}) \cdot d\mathbf{l} + \oint_C \mathbf{A} \cdot \frac{[\mathbf{V}(s+\Delta s, t) - \mathbf{V}(s,t)]}{\Delta s} (\Delta s) \\
&= \oint_c (\mathbf{V} \times \mathbf{B}) \cdot d\mathbf{l} + \oint_c (-\nabla \Phi) \cdot d\mathbf{l} + \oint_c (\mathbf{V} \cdot \nabla \mathbf{A}) \cdot d\mathbf{l} + \oint_c \mathbf{A} \cdot [d\mathbf{l} \cdot (\nabla \mathbf{V})] \\
&= \oint_c [\mathbf{V} \times (\nabla \times \mathbf{A})] \cdot d\mathbf{l} - \oint_c d\Phi + \oint_c (\mathbf{V} \cdot \nabla \mathbf{A}) \cdot d\mathbf{l} + \oint_c d\mathbf{l} \cdot [\nabla(\mathbf{A}^c \cdot \mathbf{V})] \\
&= \oint_c \{-\mathbf{V} \cdot \nabla \mathbf{A} + [\nabla(\mathbf{A} \cdot \mathbf{V}^c)]\} \cdot d\mathbf{l} + \oint_c (\mathbf{V} \cdot \nabla \mathbf{A}) \cdot d\mathbf{l} + \oint_c d\mathbf{l} \cdot [\nabla(\mathbf{A}^c \cdot \mathbf{V})] \\
&= \oint_c [d\mathbf{l} \cdot \nabla(\mathbf{A} \cdot \mathbf{V}^c)] + \oint_c d\mathbf{l} \cdot [\nabla(\mathbf{A}^c \cdot \mathbf{V})] \\
&= \oint_c d\mathbf{l} \cdot \nabla(\mathbf{A} \cdot \mathbf{V}) \\
&= \oint_c d(\mathbf{A} \cdot \mathbf{V}) \\
&= 0
\end{aligned} \tag{4.13}$$

Since, Eq. (4.13) is consistent with Eq. (4.1), we have successfully proved that the magnetic flux is frozen in MHD plasma.

4.3. Conservation of Circulation vs. Frozen-in Flux in MHD Plasma

The idea of frozen-in flux of MHD plasma is adopt from conservation of circulation in an ideal fluid, where we define an ideal fluid is a non-viscous and isentropic fluid. (e.g., Landau and Lifshitz (Fluid Mechanics, 2nd ed. 1989)

Momentum equation of a non-viscous fluid is

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\frac{\nabla p}{\rho} + \mathbf{g} \tag{4.14a}$$

or

$$\frac{\partial \mathbf{V}}{\partial t} - \mathbf{V} \times (\nabla \times \mathbf{V}) + \nabla \frac{V^2}{2} = -\frac{\nabla p}{\rho} - \nabla \Phi_g \tag{4.14b}$$

Vorticity equation of a non-viscous fluid can be obtained from curl of the momentum equation (4.14b)

$$\frac{\partial \nabla \times \mathbf{V}}{\partial t} - \nabla \times [\mathbf{V} \times (\nabla \times \mathbf{V})] + \nabla \times [\nabla \frac{V^2}{2}] = -\nabla \times (\frac{\nabla p}{\rho}) - \nabla \times \nabla \Phi_g$$

Since $\nabla \times \nabla f = 0$, the above equation can be simplified as

$$\boxed{\frac{\partial \mathbf{\Omega}}{\partial t} - \nabla \times [\mathbf{V} \times \mathbf{\Omega}] = \frac{\nabla \rho \times \nabla p}{\rho^2}} \quad (4.15)$$

where $\mathbf{\Omega} = \nabla \times \mathbf{V}$. Eq. (4.15) is called vorticity equation.

If we consider an isentropic fluid, we have

$p\rho^{-\gamma} = \text{constant}$

or

$$\frac{\nabla p}{p} = \gamma \frac{\nabla \rho}{\rho}$$

Namely, vectors ∇p and $\nabla \rho$ are parallel to each other in an isentropic fluid. Thus, $\nabla \rho \times \nabla p = 0$. The vorticity equation (4.15) becomes

$$\boxed{\frac{\partial \mathbf{\Omega}}{\partial t} - \nabla \times [\mathbf{V} \times \mathbf{\Omega}] = 0} \quad (4.16)$$

Since Eq. (4.16) is similar to the combination of MHD Ohm's law and Faraday's law, i.e.,

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} - \nabla \times [\mathbf{V} \times \mathbf{B}] = 0} \quad (4.17)$$

and since both where $\mathbf{\Omega}$ and $\mathbf{B}$ are divergent-free vectors, we can follow the similar procedure as described in section 4.2 to show that circulation $\Gamma \equiv \oint \mathbf{V} \cdot d\mathbf{l}$ is conserved along the path line of an ideal fluid element. Namely,

$$\boxed{\frac{d\Gamma}{dt} = 0 = \frac{d}{dt} \oint \mathbf{V} \cdot d\mathbf{l} = \frac{d}{dt} \iint (\nabla \times \mathbf{V}) \cdot d\mathbf{a} = \frac{d}{dt} \iint \mathbf{\Omega} \cdot d\mathbf{a}} \quad (4.18)$$

This is called *conservation of circulation* in an ideal fluid.

Exercise 4.2.

(a) Show that $\nabla \times \nabla f = 0$

(b) Show that $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.

Answer of Exercise 4.2(a):

Consider the following integration

$$\iint_{S(L)} \nabla \times \nabla f = \oint_L (\nabla f) \cdot d\mathbf{l} = \oint_L \frac{df}{dl} dl = \oint_L df = 0$$

Thus, $\nabla \times \nabla f = 0$.

Answer of Exercise 4.2(b):

Consider the following integration

$$\iiint_{V(S)} \nabla \cdot (\nabla \times \mathbf{A}) = \oint\!\!\!\oint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a}$$

Cut the closed surface S into two parts S_1 and S_2 as shown in Figure 4.2. Let the cutting edge form a closed loop L. Let $S = S_1(L) + S_2(-L)$. Thus, the above integration can be written as

$$\iiint_{V(S)} \nabla \cdot (\nabla \times \mathbf{A}) = \oint\!\!\!\oint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a}$$

$$= \iint_{S_1(L)} (\nabla \times \mathbf{A}) \cdot d\mathbf{a} + \iint_{S_2(-L)} (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint_L \mathbf{A} \cdot d\mathbf{l} - \oint_L \mathbf{A} \cdot d\mathbf{l} = 0$$

Thus, $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.

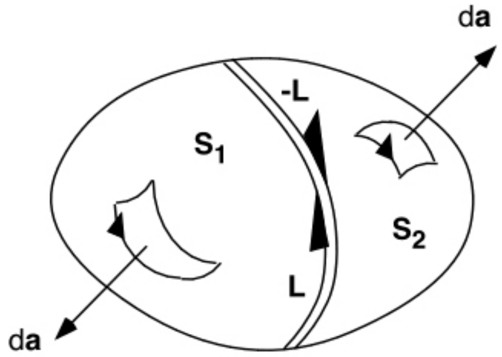

Figure 4.2. Sketches of how to cut the closed surface S into two parts S_1 and S_2 with cutting edge at loop L.