Lecture 6. Linear Waves in Magnetohydrodynamic Plasma

6.0. How to Linearize a Nonlinear Equation

We shall use the mass continuity equation as an example to demonstrate how to linearize a nonlinear equation. Let A_0 denotes a background state and A_1 denotes a small perturbation, where $O(A_1) = O(\varepsilon)O(A_0)$. Then, A can be written as

$$A = A_0 + A_1 + O(\varepsilon^2)O(A_0) \approx A_0 + A_1 \tag{6.0.1}$$

Substituting equation (6.0.1) into the mass continuity equation, it yields

$$[\frac{\partial}{\partial t} + (\mathbf{V}_0 + \mathbf{V}_1)\cdot\nabla](\rho_0 + \rho_1) = -(\rho_0 + \rho_1)\nabla\cdot(\mathbf{V}_0 + \mathbf{V}_1) \tag{6.0.2}$$

The equilibrium state of continuity equation is

$$(\mathbf{V}_0\cdot\nabla)\rho_0 = -\rho_0\nabla\cdot\mathbf{V}_0 \tag{6.0.3}$$

Subtracting equation (6.0.3) from equation (6.0.2) yields

$$(\frac{\partial}{\partial t} + \mathbf{V}_0\cdot\nabla)\rho_1 + \mathbf{V}_1\cdot\nabla\rho_0 + \mathbf{V}_1\cdot\nabla\rho_1 = -\rho_0\nabla\cdot\mathbf{V}_1 - \rho_1\nabla\cdot\mathbf{V}_0 - \rho_1\nabla\cdot\mathbf{V}_1 \tag{6.0.4}$$

where $\mathbf{V}_1\cdot\nabla\rho_1$ and $\rho_1\nabla\cdot\mathbf{V}_1$ are of the order of $O(\varepsilon^2)$. Ignoring these nonlinear second-order small terms, equation (6.0.4) is reduced to a linearized equation,

$$(\frac{\partial}{\partial t} + \mathbf{V}_0\cdot\nabla)\rho_1 + \mathbf{V}_1\cdot\nabla\rho_0 = -\rho_0\nabla\cdot\mathbf{V}_1 - \rho_1\nabla\cdot\mathbf{V}_0 \tag{6.0.5}$$

The linearized equation shown in equation (6.0.5) can be used to study linear waves in a non-uniform background medium with either density gradient or velocity shear.

6.1. Linear Plane Waves in Uniform MHD Plasma

Magnetohydrodynamic (MHD) plasma is a plasma model under long wavelength and low frequency limit, in which the time scale and spatial scale of the MHD plasma phenomena are much longer than the ions' time scale and spatial scale, respectively. Lecture 4 shows that the MHD Ohm's law can lead to frozen-in flux, which is an important characteristic of MHD plasma. In addition to the characteristics of frozen-in conditions, MHD linear wave modes are also important characteristics of the MHD plasma. Governing equations of MHD plasma with isotropic pressure and zero heat flux are listed in Column (1) of Table 6.1.

Table 6.1. Governing equations of MHD plasma with isotropic pressure and zero heat flux

(1) MHD equations in (t, $\mathbf{x}$) domain	(2) linearized MHD equations in ($\omega, \mathbf{k}$) domain
Mass continuity equation $(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla)\rho = -\rho \nabla \cdot \mathbf{V}$	Mass continuity equation $(-i\omega)\tilde{\rho}_1 = -\rho_0 (i\mathbf{k}) \cdot \tilde{\mathbf{V}}_1$ (6.1)
MHD momentum equation $\rho(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla p + \mathbf{J} \times \mathbf{B}$	MHD momentum equation $\rho_0(-i\omega)\tilde{\mathbf{V}}_1 = -(i\mathbf{k})\tilde{p}_1 + \tilde{\mathbf{J}}_1 \times \mathbf{B}_0$ (6.2)
MHD energy equation $\frac{3}{2}[(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla)\ln(p\rho^{-5/3})] = 0$	MHD energy equation $(-i\omega)\tilde{p}_1 = \frac{\gamma p_0}{\rho_0}(-i\omega)\tilde{\rho}_1$ (6.3)
MHD charge continuity equation $\nabla \cdot \mathbf{J} = 0$	MHD charge continuity equation $(i\mathbf{k}) \cdot \tilde{\mathbf{J}}_1 = 0$ (6.4)
MHD Ohm's law $\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$	MHD Ohm's law $\tilde{\mathbf{E}}_1 + \tilde{\mathbf{V}}_1 \times \mathbf{B}_0 = 0$ (6.5)
Maxwell's equations: $\nabla \cdot \mathbf{E} = 0$ $\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$	Maxwell's equations: $(i\mathbf{k}) \cdot \tilde{\mathbf{E}}_1 = 0$ (6.6) $(i\mathbf{k}) \cdot \tilde{\mathbf{B}}_1 = 0$ (6.7) $(i\mathbf{k}) \times \tilde{\mathbf{E}}_1 = i\omega\tilde{\mathbf{B}}_1$ (6.8) $(i\mathbf{k}) \times \tilde{\mathbf{B}}_1 = \mu_0 \tilde{\mathbf{J}}_1$ (6.9)

For uniform background plasma, we can choose a moving frame such that $\mathbf{V}_0 = 0$. Substituting $\mathbf{V}_0 = 0$ into Ohm's law, it yields $\mathbf{E}_0 = 0$. Far from the source region, perturbations can be assumed in plane-wave format. A perturbation $A_1(\mathbf{x}, t)$ can be written as

$$A_1(\mathbf{x}, t) = \overline{A}_1(\mathbf{k}, \omega)\cos(\mathbf{k} \cdot \mathbf{x} - \omega t + \phi_A) = \mathrm{Re}\{\tilde{A}_1(\mathbf{k}, \omega)\exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]\}$$

where $\tilde{A}_1(\mathbf{k}, \omega) = \overline{A}_1(\mathbf{k}, \omega)e^{i\phi_A}$ is a complex number. The wave amplitude $\overline{A}_1(\mathbf{k}, \omega)$ satisfies $O(\overline{A}_1) = O(\varepsilon)O(A_0)$, where A_0 denotes a background variable. Following the procedures described in equations (6.0.1)~(6.0.5), a set of linearized MHD equations in ($\omega, \mathbf{k}$) domain are obtained and are listed in Column (2) of Table 6.1.

Our goal is to reduce the system equations listed in Table 6.1 Column (2) into a set of equations for plasma flow velocity $\tilde{\mathbf{V}}_1$. We shall focus on the momentum equation (6.2). In order to eliminate $\tilde{p}_1$ in Eq. (6.2), we substitute Eq. (6.1) into Eq. (6.3) to eliminate $\tilde{\rho}_1$, then substitute the resulting equation into Eq. (6.2) to eliminate $\tilde{p}_1$. Likewise, to eliminate $\tilde{\mathbf{J}}_1$ in Eq. (6.2), we can substitute Eq. (6.5) into Eq. (6.8) to eliminate $\tilde{\mathbf{E}}_1$, then substitute the resulting equation into Eq. (6.9) to eliminate $\tilde{\mathbf{B}}_1$, and then substitute the resulting equation into Eq. (6.2) to eliminate $\tilde{\mathbf{J}}_1$.

Substituting Eq. (6.1) into Eq. (6.3) yields

$$\tilde{p}_1 = \frac{\gamma p_0}{\rho_0}\tilde{\rho}_1 = C_{S0}^2\tilde{\rho}_1 = C_{S0}^2\frac{\rho_0 \mathbf{k}\cdot\tilde{\mathbf{V}}_1}{\omega} \quad (6.3')$$

Substituting Eq. (6.5) into Eq. (6.8) to eliminate $\tilde{\mathbf{E}}_1$, then substituting the resulting equation into Eq. (6.9) to eliminate $\tilde{\mathbf{B}}_1$, it yields

$$\tilde{\mathbf{J}}_1 = \frac{i\mathbf{k}\times\tilde{\mathbf{B}}_1}{\mu_0} = \frac{i\mathbf{k}\times\frac{\mathbf{k}\times\tilde{\mathbf{E}}_1}{\omega}}{\mu_0} = \frac{i\mathbf{k}\times\frac{\mathbf{k}\times(-\tilde{\mathbf{V}}_1\times\mathbf{B}_0)}{\omega}}{\mu_0} = \frac{i\mathbf{k}\times[\mathbf{k}\times(\mathbf{B}_0\times\tilde{\mathbf{V}}_1)]}{\mu_0\omega} \quad (6.9')$$

Substituting Eqs. (6.3') and (6.9') into Eq. (6.2) yields

$$\rho_0(-i\omega)\tilde{\mathbf{V}}_1 = -i\mathbf{k}C_{S0}^2\frac{\rho_0\mathbf{k}\cdot\tilde{\mathbf{V}}_1}{\omega} + \frac{i\mathbf{k}\times[\mathbf{k}\times(\mathbf{B}_0\times\tilde{\mathbf{V}}_1)]}{\mu_0\omega}\times\mathbf{B}_0 \quad (6.2')$$

Multiplying Eq. (6.2') by $i\omega/\rho_0 k^2$ yields

$$\frac{\omega^2}{k^2}\tilde{\mathbf{V}}_1 = C_{S0}^2\hat{k}\hat{k}\cdot\tilde{\mathbf{V}}_1 + C_{A0}^2\hat{B}_0\times\{\hat{k}\times[\hat{k}\times(\hat{B}_0\times\tilde{\mathbf{V}}_1)\}$$

where $C_{A0}\equiv B_0/\sqrt{\mu_0\rho_0}$ is called Alfvén speed, and $C_{S0}\equiv\sqrt{\gamma p_0/\rho_0}$ is called sound speed.

As a result, we can obtain a set of equations for flow velocity $\tilde{\mathbf{V}}_1$, which can be written as

$$\mathsf{D}\cdot\tilde{\mathbf{V}}_1 = 0 \quad (6.10)$$

where

$$\mathsf{D} = [\frac{\omega^2}{k^2} - C_{A0}^2(\hat{B}_0\cdot\hat{k})^2]\mathbf{1} - (C_{A0}^2 + C_{S0}^2)\hat{k}\hat{k} + C_{A0}^2(\hat{B}_0\cdot\hat{k})(\hat{B}_0\hat{k} + \hat{k}\hat{B}_0) \quad (6.11)$$

For convenience, we can choose a coordinate system such that background magnetic field is along the $\hat{z}$-axis, and wave number $\mathbf{k}$ lies on x- z plane. Namely,

$$\mathbf{B}_0 = \hat{z} B_0 \tag{6.12}$$

and

$$\mathbf{k} = k(\hat{z}\cos\theta + \hat{x}\sin\theta) \tag{6.13}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{B}_0$. Substituting Eqs. (6.12) and (6.13) into Eqs. (6.10) and (6.11) yields

$$\begin{pmatrix} (\omega^2/k^2)-\alpha & 0 & -\delta \\ 0 & (\omega^2/k^2)-C_{A0}^2\cos^2\theta & 0 \\ -\delta & 0 & (\omega^2/k^2)-\beta \end{pmatrix} \begin{pmatrix} V_{1x} \\ V_{1y} \\ V_{1z} \end{pmatrix} = 0 \tag{6.14}$$

where

$$\alpha = C_{A0}^2\cos^2\theta + (C_{A0}^2 + C_{S0}^2)\sin^2\theta = C_{A0}^2 + C_{S0}^2\sin^2\theta \tag{6.15}$$

$$\beta = C_{A0}^2\cos^2\theta + (C_{A0}^2 + C_{S0}^2)\cos^2\theta - 2C_{A0}^2\cos^2\theta = C_{S0}^2\cos^2\theta \tag{6.16}$$

$$\delta = C_{S0}^2\cos\theta\sin\theta \tag{6.17}$$

Note that solutions of ω^2/k^2 for different wave modes can be considered as eigen values of the following matrix

$$\begin{pmatrix} \alpha & 0 & \delta \\ 0 & C_{A0}^2\cos^2\theta & 0 \\ \delta & 0 & \beta \end{pmatrix}$$

Characteristics of different wave modes can be obtained from the corresponding eigen vectors.

Exercise 6.1.

Review eigen values and eigen vectors of a symmetric matrix. Determine eigen values λ_1, λ_2, λ_3, and the corresponding normalized eigen vectors $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$, of the following symmetric matrix

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Show that these eigen vectors of the symmetric matrix form an orthonormal basis and after coordinate transformation, the representation of matrix M in this new basis $B' = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ becomes

$$M = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}_{B'}$$

6.2. Linear Wave Modes in the MHD Plasma

Number of linearized equations with time derivative term can lead to the same number of linear wave modes. There are seven equations in Table 6.1 that consist of a time derivative term. It will be shown in this section that, for $\theta \neq 0$ and $\theta \neq \pi/2$, seven linear wave modes can be found in MHD plasma. Three of them are forward propagating waves. Based on their wave speeds, these three wave modes are called fast-mode wave, intermediate-mode wave, and slow-mode wave. The intermediate mode wave is also called Alfvén-mode wave or share-Alfvén wave. The other four wave modes are backward propagating fast-mode wave, intermediate-mode wave, slow-mode wave, and non-propagating entropy-mode wave. The fast mode, Alfvén mode, and slow mode are eigen modes of Eq. (6.14). The entropy mode is an additional wave mode, which can be obtained from equation of ρ_1 (i.e., continuity equation).

6.2.1. Entropy Mode

Entropy mode in MHD plasma is characterized by $\rho_1 \neq 0$, but $V_{1x} = V_{1y} = V_{1z} = 0$ and $\omega = 0$. For $\omega = 0$, the phase speed also vanishes. Thus, entropy mode is frozen in the plasma flow.

In general, if $V_{1x} = V_{1y} = V_{1z} = 0$, but $\rho_1 \neq 0$ and/or $\mathbf{B}_1 \neq 0$, then ω must be zero ($\omega = 0$), and $-(i\mathbf{k})\tilde{p}_1 + \tilde{\mathbf{J}}_1 \times \mathbf{B}_0 = 0$.

Proof:

For $V_{1x} = V_{1y} = V_{1z} = 0$, Eq. (6.10) or (6.14) is automatically fulfilled.

Substituting $\mathbf{V}_1 = 0$ into Eq. (6.5) yields $\mathbf{E}_1 = 0$.

Substituting $\mathbf{V}_1 = 0$ into Eq. (6.1) yields $\omega\tilde{\rho}_1 = 0$.

Substituting $\mathbf{E}_1 = 0$ into Eq. (6.8) yields $\omega\tilde{\mathbf{B}}_1 = 0$.

Thus, if $\rho_1 \neq 0$ and/or $\mathbf{B}_1 \neq 0$, then we must have $\omega = 0$.

Substituting $\mathbf{V}_1 = 0$ into Eq. (6.2) yields

$$-(i\mathbf{k})\tilde{p}_1 + \tilde{\mathbf{J}}_1 \times \mathbf{B}_0 = 0 \tag{6.2a}$$

Substituting Eq. (6.9) into Eq. (6.2a) yields

$$-\mathbf{k}\tilde{p}_1 - \frac{\mathbf{k}(\tilde{\mathbf{B}}_1 \cdot \mathbf{B}_0)}{\mu_0} + \frac{(\mathbf{k}\cdot\mathbf{B}_0)\tilde{\mathbf{B}}_1}{\mu_0} = 0 \tag{6.2b}$$

Eq. (6.7) implies $\mathbf{B}_1 \perp \mathbf{k}$, thus Eq. (6.2b) can be decomposed into two components. One of them is in $\mathbf{k}$ direction. The other is in $\mathbf{B}_1$ direction. That is

$$-\mathbf{k}(\tilde{p}_1 + \frac{\tilde{\mathbf{B}}_1 \cdot \mathbf{B}_0}{\mu_0}) = 0 \tag{6.2c}$$

and

$$(\mathbf{k}\cdot\mathbf{B}_0)\tilde{\mathbf{B}}_1 = 0 \tag{6.2d}$$

Eq. (6.2d) implies if $\mathbf{B}_1 \neq 0$ then $\mathbf{k}\cdot\mathbf{B}_0 = 0$. Likewise, if $\mathbf{k}\cdot\mathbf{B}_0 \neq 0$ then $\mathbf{B}_1 = 0$.
Thus, solutions of $\omega = 0$ can be classified into the following types:

If $\mathbf{B}_1 \neq 0$, $\rho_1 = p_1 = 0$, and $\mathbf{k}\cdot\mathbf{B}_0 = 0$, then wave mode with $\omega = 0$ can be considered as perpendicular-propagated Alfvén-mode wave . Eq. (6.2c) yields $\mathbf{B}_1 \cdot \mathbf{B}_0 = 0$ in this case.

If $\mathbf{B}_1 \neq 0$, $p_1 \neq 0$, and $\mathbf{k}\cdot\mathbf{B}_0 = 0$, then wave mode with $\omega = 0$ can be considered as perpendicular-propagated slow-mode wave. Eq. (6.2c) yields $\mathbf{B}_1 \cdot \mathbf{B}_0 \neq 0$ in this case.

If $\omega = 0$, $\rho_1 \neq 0$ and $\mathbf{k}\cdot\mathbf{B}_0 \neq 0$, then Eq. (6.2d) and (6.2c) yield $p_1 = 0$ and $\mathbf{B}_1 = 0$. This wave mode is called entropy mode. Note that for $\omega = 0$, Eq. (6.3) is automatically fulfilled.

It can be shown that solutions of nonlinear MHD equilibrium states consist of Contact Discontinuity (CD), Tangential Discontinuity (TD), Rotational Discontinuity (RD), and Shock Waves. (e.g., Kantrowitz and Petschek, 1966; and Chao, 1970. Or see Chapter 2 in my lecture notes of Nonlinear Space Plasma Physics.)

It can be shown that Tangential Discontinuity (TD) can be considered as a nonlinear version of perpendicularly propagated Alfvén-mode wave or slow-mode wave. Contact Discontinuity (CD) can be considered as a nonlinear version of entropy-mode wave in MHD plasma.

6.2.2. Alfvén Mode (or Intermediate Mode)

Alfvén mode in MHD plasma is characterized by $V_{1x} = V_{1z} = 0$ but $V_{1y} \neq 0$. For $V_{1x} = V_{1z} = 0$ but $V_{1y} \neq 0$, Eq. (6.14) yields

$$\frac{\omega^2}{k^2} = C_{A0}^2 \cos^2\theta \tag{6.18}$$

Eq. (6.18) is the wave dispersion relation of Alfvén-mode wave. Since the phase speed of Alfvén mode is in between fast-mode and slow-mode wave speed, the Alfvén mode is also called intermediate mode. It can be shown that Rotational Discontinuity (RD) can be considered as a nonlinear version of Alfvén-mode wave in MHD plasma.

Characteristics of Alfvén-mode wave:

From Alfvén-mode wave dispersion relation $\omega = \pm k C_{A0} \cos\theta$, we can determine group velocity of Alfvén mode to be

$$\mathbf{v}_g = \frac{d\omega}{d\mathbf{k}} = \hat{x}\frac{\partial\omega}{\partial k_x} + \hat{z}\frac{\partial\omega}{\partial k_z} = \pm\hat{z}C_{A0} = \pm\hat{B}_0 C_{A0}$$

Exercise 6.2.

(1) Show that for Alfvén wave $\rho_1 = 0$, $p_1 = 0$, and $B_1 = 0$. Show that B_1 can be determined from $B_1 = \mathbf{B}_1 \cdot \hat{B}_0$.

(2) Determine perturbation directions of $\mathbf{V}_1$, $\mathbf{E}_1$, $\mathbf{B}_1$, and $\mathbf{J}_1$ for Alfvén-mode wave.

(3) Determine relationship between $\mathbf{B}_1$ and $\mathbf{V}_1$ in Alfvén-mode wave. Show that variations of $\mathbf{B}_1$ and $\mathbf{V}_1$ are in phase if $\pi/2 < \theta < \pi$, but out-off phase if $0 < \theta < \pi/2$.

6.2.3. Fast Mode and Slow Mode

For $V_{1y} = 0$ but $\begin{pmatrix} V_{1x} \\ V_{1z} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Eq. (6.14) yields

$$\det\begin{pmatrix} (\omega^2/k^2)-\alpha & -\delta \\ -\delta & (\omega^2/k^2)-\beta \end{pmatrix} = \left(\frac{\omega^2}{k^2}\right)^2 - \frac{\omega^2}{k^2}(\alpha+\beta)+\alpha\beta-\delta^2 = 0$$

where α, β, and δ are given in Eqs. (6.15)~(6.17), which yields

$$\alpha+\beta = C_{A0}^2 + C_{S0}^2\sin^2\theta + C_{S0}^2\cos^2\theta = C_{A0}^2 + C_{S0}^2$$

and

$$\alpha\beta-\delta^2 = (C_{A0}^2 + C_{S0}^2\sin^2\theta)C_{S0}^2\cos^2\theta - C_{S0}^4\cos^2\theta\sin^2\theta = C_{A0}^2 C_{S0}^2\cos^2\theta$$

Thus, we have

$$\left(\frac{\omega^2}{k^2}\right)^2 - \frac{\omega^2}{k^2}(C_{A0}^2 + C_{S0}^2) + C_{A0}^2 C_{S0}^2\cos^2\theta = 0 \qquad (6.20)$$

Eq. (6.20) has two roots of ω^2/k^2. They are the fast-mode (+) and slow-mode (–) dispersion relation

$$\left(\frac{\omega^2}{k^2}\right)_{\substack{Fast\\Slow}} = (v_{ph}^2)_{\substack{Fast\\Slow}} = \frac{1}{2}\left\{(C_{A0}^2 + C_{S0}^2) \pm \sqrt{(C_{A0}^2 + C_{S0}^2)^2 - 4C_{A0}^2 C_{S0}^2\cos^2\theta}\right\} \qquad (6.21)$$

Characteristics of Fast-mode and Slow-mode waves:

From Fast-mode and Slow-mode wave dispersion relation, we can determine group velocity of these two wave modes as

$$(\mathbf{v}_g)_{\substack{Fast\\Slow}} = \frac{\omega}{k}\left(\hat{k}\frac{\partial\omega}{\partial k} + \hat{\theta}\frac{1}{k}\frac{\partial\omega}{\partial\theta}\right) = \hat{k}(v_{ph})_{\substack{Fast\\Slow}} \pm \hat{\theta}\frac{1}{(v_{ph})_{\substack{Fast\\Slow}}}\frac{C_{A0}^2 C_{S0}^2\cos\theta\sin\theta}{\sqrt{(C_{A0}^2 + C_{S0}^2)^2 - 4C_{A0}^2 C_{S0}^2\cos^2\theta}} \qquad (6.22)$$

where $(v_{ph})_{\substack{Fast\\Slow}}$ is given in Eq. (6.21).

Proof of Eq. (6.22):

By definition, group velocity is

$$\mathbf{v}_g = \frac{d\omega}{d\mathbf{k}} = \hat{k}\frac{\partial\omega}{\partial k} + \hat{\theta}\frac{1}{k}\frac{\partial\omega}{\partial\theta}$$

where

$$2\omega\frac{\partial\omega}{\partial k} = 2k\frac{\omega^2}{k^2}$$

and

$$2\omega\frac{\partial\omega}{\partial\theta} = k^2\frac{\partial}{\partial\theta}[\frac{1}{2}\{(C_{A0}^2 + C_{S0}^2) \pm \sqrt{(C_{A0}^2 + C_{S0}^2)^2 - 4C_{A0}^2C_{S0}^2\cos^2\theta}\}]$$

$$= k^2(\frac{1}{2})(\pm\frac{1}{2})\frac{4\cdot 2C_{A0}^2C_{S0}^2\cos\theta\sin\theta}{\sqrt{(C_{A0}^2 + C_{S0}^2)^2 - 4C_{A0}^2C_{S0}^2\cos^2\theta}}$$

Thus, we have

$$v_{ph}\mathbf{v}_g = \frac{\omega}{k}(\hat{k}\frac{\partial\omega}{\partial k} + \hat{\theta}\frac{1}{k}\frac{\partial\omega}{\partial\theta}) = \hat{k}\frac{\omega^2}{k^2} \pm \hat{\theta}\frac{C_{A0}^2C_{S0}^2\cos\theta\sin\theta}{\sqrt{(C_{A0}^2 + C_{S0}^2)^2 - 4C_{A0}^2C_{S0}^2\cos^2\theta}}$$

$$= \hat{k}[\frac{1}{2}\{(C_{A0}^2 + C_{S0}^2) \pm \sqrt{(C_{A0}^2 + C_{S0}^2)^2 - 4C_{A0}^2C_{S0}^2\cos^2\theta}\}] \pm \hat{\theta}\frac{C_{A0}^2C_{S0}^2\cos\theta\sin\theta}{\sqrt{(C_{A0}^2 + C_{S0}^2)^2 - 4C_{A0}^2C_{S0}^2\cos^2\theta}}$$

or

$$(\mathbf{v}_g)_{\substack{Fast\\Slow}} = \frac{\omega}{k}(\hat{k}\frac{\partial\omega}{\partial k} + \hat{\theta}\frac{1}{k}\frac{\partial\omega}{\partial\theta}) = \hat{k}(v_{ph})_{\substack{Fast\\Slow}} \pm \hat{\theta}\frac{1}{(v_{ph})_{\substack{Fast\\Slow}}}\frac{C_{A0}^2C_{S0}^2\cos\theta\sin\theta}{\sqrt{(C_{A0}^2 + C_{S0}^2)^2 - 4C_{A0}^2C_{S0}^2\cos^2\theta}}$$

Exercise 6.3.

(1) Determine phase relationship of ρ_1 and B_1, for fast-mode and slow-mode waves.

(2) Determine perturbation directions of $\mathbf{V}_1$, $\mathbf{E}_1$, $\mathbf{B}_1$, and $\mathbf{J}_1$ for fast-mode and slow-mode waves.

(3) Show that $\mathbf{V}_{1Fast}\cdot\mathbf{V}_{1Slow} = 0$.

Proof of Exercise 6.3(3) $\mathbf{V}_{1Fast}\cdot\mathbf{V}_{1Slow} = 0$

Eq. (6.14) yields

$$(V_{1x})_{Fast}[(\omega^2/k^2)_{Fast} - \alpha] - (V_{1z})_{Fast}\delta = 0$$

and

$$(V_{1x})_{Slow}[(\omega^2/k^2)_{Slow} - \alpha] - (V_{1z})_{Slow}\delta = 0$$

Substituting the above two equations into $\mathbf{V}_{1Fast}\cdot\mathbf{V}_{1Slow}$, it yields

$\mathbf{V}_{1Fast} \cdot \mathbf{V}_{1Slow} = (V_{1x})_{Fast}(V_{1x})_{Slow} + (V_{1z})_{Fast}(V_{1z})_{Slow}$

$= (V_{1x})_{Fast}(V_{1x})_{Slow} + \{(V_{1x})_{Fast}[(\omega^2/k^2)_{Fast} - \alpha]/\delta\}\{(V_{1x})_{Slow}[(\omega^2/k^2)_{Slow} - \alpha]/\delta\}$

$= (V_{1x})_{Fast}(V_{1x})_{Slow}\{1 + [(\omega^2/k^2)_{Fast} - \alpha][(\omega^2/k^2)_{Slow} - \alpha]/\delta^2\}$

$= (V_{1x})_{Fast}(V_{1x})_{Slow}\{\delta^2 + (\omega^2/k^2)_{Fast}(\omega^2/k^2)_{Slow} - \alpha[(\omega^2/k^2)_{Fast} + (\omega^2/k^2)_{Slow}] + \alpha^2\}/\delta^2$

$= (V_{1x})_{Fast}(V_{1x})_{Slow}\{\delta^2 + \alpha^2 + \frac{1}{4}[b^2 - (b^2 - 4c)] - \alpha b\}/\delta^2$

$= (V_{1x})_{Fast}(V_{1x})_{Slow}\{\delta^2 + \alpha^2 + c - \alpha b\}/\delta^2$

where

$b = (\alpha + \beta)$

$c = \alpha\beta - \delta^2$

Thus

$\mathbf{V}_{1Fast} \cdot \mathbf{V}_{1Slow} = (V_{1x})_{Fast}(V_{1x})_{Slow}\{\delta^2 + \alpha^2 + (\alpha\beta - \delta^2) - \alpha(\alpha + \beta)\}/\delta^2 = 0$

Proof of (1)

(6.1): $(-i\omega)\tilde{\rho}_1 = -\rho_0(i\mathbf{k}) \cdot \tilde{\mathbf{V}}_1$

(6.3'): $\tilde{p}_1 = \frac{\gamma p_0}{\rho_0}\tilde{\rho}_1 = C_{S0}^2\tilde{\rho}_1$

(6.2): $\rho_0(-i\omega)\tilde{\mathbf{V}}_1 = -(i\mathbf{k})\tilde{p}_1 + \tilde{\mathbf{J}}_1 \times \mathbf{B}_0$

(6.9): $(i\mathbf{k}) \times \tilde{\mathbf{B}}_1 = \mu_0\tilde{\mathbf{J}}_1$

Substituting (6.9) into (6.2), it yields

$$\rho_0(-i\omega)\tilde{\mathbf{V}}_1 = -(i\mathbf{k})\tilde{p}_1 + (i\mathbf{k} \times \tilde{\mathbf{B}}_1) \times \mathbf{B}_0 / \mu_0 \quad (6.2')$$

Substituting (6.3') into $\mathbf{k} \cdot$(6.2') to eliminate $\tilde{\mathbf{J}}_1$, then substituting (6.1) into the resulting equation to eliminate $\tilde{\mathbf{V}}_1$ and substituting (6.3') into the resulting equation to eliminate $\tilde{p}_1$, it yields

$$\rho_0(-i\omega)\mathbf{k} \cdot \tilde{\mathbf{V}}_1 = -\mathbf{k} \cdot (i\mathbf{k})\tilde{p}_1 + \mathbf{k} \cdot [(i\mathbf{k} \times \tilde{\mathbf{B}}_1) \times \mathbf{B}_0]/\mu_0$$

$$\Rightarrow (-i\omega^2)\tilde{\rho}_1 = -ik^2C_{S0}^2\tilde{\rho}_1 + i\frac{\mathbf{k} \cdot \mathbf{B}_0\mathbf{k} \cdot \tilde{\mathbf{B}}_1}{\mu_0} - ik^2\frac{\mathbf{B}_0 \cdot \tilde{\mathbf{B}}_1}{\mu_0} \quad (6.2'')$$

where $\mathbf{k} \cdot \tilde{\mathbf{B}}_1 = 0$. It can be shown that $B - B_0 = B_1 = \mathbf{B}_1 \cdot (\mathbf{B}_0 / B_0)$

Thus, the above equation (6.2") can be rewritten as

$$(\omega^2)\frac{\tilde{\rho}_1}{\rho_0} = k^2C_{S0}^2\frac{\tilde{\rho}_1}{\rho_0} + k^2\frac{B_0^2}{\rho_0\mu_0}\frac{\tilde{B}_1}{B_0} = k^2C_{S0}^2\frac{\tilde{\rho}_1}{\rho_0} + k^2C_{A0}^2\frac{\tilde{B}_1}{B_0}$$

$$\Rightarrow (\frac{\omega^2}{k^2} - C_{S0}^2)\frac{\tilde{\rho}_1}{\rho_0} = C_{A0}^2 \frac{\tilde{B}_1}{B_0} \quad (6.2''')$$

Thus, for $\omega^2/k^2 > C_{S0}^2$, variations of ρ_1 and B_1 are in phase.

For $\omega^2/k^2 < C_{S0}^2$, variations of ρ_1 and B_1 are out-off phase.

It can be shown that, for fast-mode wave, we have $(\omega^2/k^2)_{Fast} \geq C_{S0}^2$. Thus, for fast-mode wave, variations of ρ_1 and B_1 are in phase. For slow-mode wave, we have $(\omega^2/k^2)_{Slow} \leq C_{S0}^2$. Thus, for slow-mode wave, variations of ρ_1 and B_1 are out-off phase.

Note that (6.3') yields variations of ρ_1 and p_1 are in phase. Equation (6.2''') can be rewritten as

$$(\frac{\omega^2}{k^2} - C_{S0}^2)\frac{\tilde{p}_1}{C_{S0}^2\rho_0} = C_{A0}^2 \frac{\tilde{B}_1}{B_0}$$

Thus, for $\omega^2/k^2 > C_{S0}^2$, variations of p_1 and B_1 are in phase.

For $\omega^2/k^2 < C_{S0}^2$, variations of p_1 and B_1 are out-off phase.

6.2.4. Friedrichs Diagrams of the Phase Velocity and Group Velocity of the MHD Waves

Exercise 6.4.

(1) Ignoring the entropy mode, plot the phase velocities of the three MHD wave modes: fast-, Alfvén-, and slow-modes, on the Friedrichs diagram, where the polar coordinate $(r, \theta) = (\omega/k, \theta_{\mathbf{k},\mathbf{B}_0})$.

(2) Ignoring the entropy mode, plot the group velocities of the three MHD wave modes: fast-, Alfvén-, and slow-modes, on the Friedrichs diagram, where the polar coordinate $(r, \theta) = (v_g, \theta_{\mathbf{v}_g\mathbf{B}_0})$.

Students are encouraged to read the classical paper written by *Kantrowitz and Petschek* (1966) for detail discussion on the MHD wave modes. The application of the group-velocity Friedrichs diagram on wave expansion near the source region can be found in the two papers by *Lai and Lyu* (2006, 2008).

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