A Theoretical Model for Cross-Scale Simulation of Collisionless Plasmas in Space

L. H. Lyu[1,2], M. Q. Chen[1,2], and W. H. Tsai[1,2]

[1]Institute of Space Science, National Central University, Chung-Li, Taiwan 320-01, R.O.C.
[2]Department of Atmospheric Science, National Central University, Chung-Li, Taiwan 320-01, R.O.C.

Abstract

An algorithm for cross-scale plasma simulation is proposed in this study. Governing equations for cross-scale plasma simulations are derived analytically and self-consistently in three time scales: the electron time scale, the ion time scale, and the macro time scale, which ranges from quasi-MHD time scale to MHD time scale. Coupling effects between different time scales are included in each set of governing equations. In our model, large-scale low-frequency simulations will provide initial and boundary conditions for small-scale high-frequency simulations. Whereas, anomalous dissipations due to wave-wave and wave-particle interactions can be obtained from small-scale high-frequency simulations at a few sampled points and will be feedback into large-scale low-frequency simulation. Numerical methods that are appropriate for this cross-scale simulation model will also be discussed.

I. Introduction

Cross-scale couplings are commonly observed in space plasmas. In most cases, free energy of disturbances comes from large-scale plasma motions and energy stored in the magnetic field, whereas, energy and momentum transports are usually accomplished by small-scale disturbances including interactions of nonlinear waves and relaxation of plasmas from non-thermal-dynamic-equilibrium states. As a result, development of cross-scale simulation model has been the up-to-date research direction in recent years.

The objective of this study is to design a theoretical model for cross-scale plasma simulation, that can not only simulate large-scale energy and momentum input and output processes efficiently, but can also handle the small-scale energy transport processes in an appropriate and self-consistent manner.

II. Formulation of Cross-Scale Plasma Equations

Governing equations for cross-scale plasma simulations can be formulated in three distinct time intervals: (1) the electron-scale time interval, which ranges approximately from $0.1\tau_e$ to $0.1\tau_i$; (2) the ion-scale time interval, which ranges approximately from $0.1\tau_i$ to $100\tau_i$; and (3) the macro-scale time interval, which ranges from quasi-MHD time scale ($100\tau_i$) to MHD time scale ($\tau >> \tau_i$), where τ_e and τ_i are the characteristic time scale for electrons and ions, respectively.

1. Basic equations for electron-time-scale plasma

Full particle simulation codes are commonly used for studying electron time scale plasma phenomena. Based on our previous experiences in electromagnetic particle code simulation at electron time scale (Chu and Lyu, 1986; Lin and Lyu, 1995), we found that simulation is more stable if all electrons follow relativistic momentum equation and when the radiation term is included in the Maxwell's equations. However, the discrete nature in particle code simulation can give rise to difficulties on handling particle fluxes at matching boundaries between small-scale simulation and large-scale simulation. We are forced to give up particle code simulation in this study. All the kinetic simulations will be carried out by directly solving Vlasov equations in phase space. To formulate the relativistic electron Vlasov equation, we first derive the Klimontovich equation (e.g., Nicholson, 1983) for relativistic electrons, then take ensemble average of the Klimontovich equation, and then assume that the ensemble average of the micro-scale wave-particle interactions can be ignored in a collisionless plasma. As a result, we can obtain the relativistic electron Vlasov equation in the following format:

$$\frac{\partial f_e{}^e}{\partial t} + \mathbf{v}(\mathbf{u}) \cdot \frac{\partial f_e{}^e}{\partial \mathbf{x}} + \frac{-e}{m_e}(\mathbf{E}^e + \mathbf{v}(\mathbf{u}) \times \mathbf{B}^e) \cdot \frac{\partial f_e{}^e}{\partial \mathbf{u}} = 0 \qquad (1^e)$$

where superscript e denotes electron time scale variables, $\mathbf{v}(\mathbf{u}) = \mathbf{u}/\gamma(u)$, and $\gamma(u) = \sqrt{1+(u/c)^2}$. Boundary conditions of (1^e) at $u \to \infty$, satisfy $f_e{}^e \to 0$ and $\partial f_e{}^e / \partial \mathbf{u} \to 0$.

Other basic equations in the electron-time-scale simulation include the Maxwell's equations,

$$\nabla \cdot \mathbf{E}^e = \rho_c{}^e / \varepsilon_\circ$$
$$\nabla \cdot \mathbf{B}^e = 0$$
$$\frac{\partial \mathbf{B}^e}{\partial t} = -\nabla \times \mathbf{E}^e \qquad (2^e)$$
$$\nabla \times \mathbf{B}^e = \mu_\circ \varepsilon_\circ \frac{\partial \mathbf{E}^e}{\partial t} + \mu_\circ \mathbf{J}^e$$

and the definition of charge density and electric current density,

$$\rho_c{}^e = e(n_i{}^i - \iiint f_e{}^e d^3u)$$
$$\mathbf{J}^e = e[n_i{}^i\mathbf{V}_i{}^i - \iiint \mathbf{v}(\mathbf{u}) f_e{}^e d^3u] \qquad (3^e)$$

where ion density and ion flow velocity can be obtained from ion-time-scale simulation and can be considered unchanged during the entire electron-time-scale simulation period (~ 0.1 τ_i).

From (1^e), (2^e), and (3^e), we can obtain governing equations for electron-time-scale simulation, which are summarized in Table 1.

2. Basic equations for ion-time-scale plasma

Hybrid simulation code with particle ions and fluid electrons are commonly used to study ion-scale plasma phenomena (e.g., Swift and Lee, 1983; Lyu and Kan, 1993). Ion kinetic effects are fully included in the hybrid code simulations but without any information from electron-scale wave-particle and wave-wave interactions. Thus, it is our goal to incorporate modulations of these high frequency interactions in our ion-time-scale plasma equations.

To avoid problems due to discrete particle flux at boundaries, ions' Vlasov equation will be used to study kinetic behavior of ions in phase space.

$$\frac{\partial f_i{}^i}{\partial t} + \mathbf{v}\cdot\frac{\partial f_i{}^i}{\partial \mathbf{x}} + \frac{+e}{m_i}(\mathbf{E}^i + \mathbf{v}\times\mathbf{B}^i)\cdot\frac{\partial f_i{}^i}{\partial \mathbf{v}} = 0 \qquad (1^i)$$

where superscript i denotes ion time scale variables. Boundary conditions of (1^i) at $v \to v_{\max} << c$, satisfy $f_i{}^i \to 0$ and $\partial f_i{}^i/\partial \mathbf{v} \to 0$.

Ion-time-scale Maxwell's equations can be obtained by taking time average of Maxwell's equations (2^e), which yields,

$$\nabla\cdot\mathbf{E}^i = \rho_c{}^i/\varepsilon_\circ$$
$$\nabla\cdot\mathbf{B}^i = 0$$
$$\frac{\partial \mathbf{B}^i}{\partial t} = -\nabla\times\mathbf{E}^i \qquad (2^i)$$
$$\nabla\times\mathbf{B}^i = \mu_\circ\varepsilon_\circ\frac{\partial \mathbf{E}^i}{\partial t} + \mu_\circ\mathbf{J}^i \approx \mu_\circ\mathbf{J}^i$$

where we have ignore the displacement current in the Ampere's Law for ion-time-scale plasma. Since the ion-time-scale charge density $\rho_c{}^i$ is very small, we shall not use the Poisson equation in (2^i) to determine ion-time-scale electric field $\mathbf{E}^i$. Taking time average of equations (3^e), yields,

$$\rho_c{}^i = e\left[n_i{}^i - \left\langle \iiint f_e{}^e d^3u\right\rangle_i\right] \approx 0$$
$$\mathbf{J}^i = e\left[n_i{}^i\mathbf{V}_i{}^i - \left\langle \iiint \mathbf{v}(\mathbf{u}) f_e{}^e d^3u\right\rangle_i\right] \qquad (3^i)$$

where notation $\left\langle A^e\right\rangle_i$ denotes $\int_0^{0.1\tau_i} A^e(t)dt/0.1\tau_i$. By definition, we have

$$n_i{}^i \equiv \iiint f_i{}^i d^3v$$
$$n_i{}^i\mathbf{V}_i{}^i \equiv \iiint \mathbf{v} f_i{}^i d^3v$$

We can also define the time average terms in (3^i) to be

$$n_e{}^i \equiv \left\langle \iiint f_e{}^e d^3u\right\rangle_i$$
$$\mathbf{V}_e{}^i \equiv \left\langle \iiint \mathbf{v}(\mathbf{u}) f_e{}^e d^3u\right\rangle_i / n_e{}^i$$

Thus, from (3^i), we obtain ion-time-scale electron number density and flow velocity directly from ion-time-scale simulation, i.e.,

$$n_e{}^i \approx n_i{}^i \approx n^i$$
$$\mathbf{V}_e{}^i = \mathbf{V}_i{}^i - \frac{\mathbf{J}^i}{en^i} \qquad (4)$$

where $\mathbf{J}^i$ can be determined from magnetic field based on the Ampere's law. For simplicity, from now on, we shall use n^i to denote both ion number density and electron number density in ion time scale formulation.

The basic equations in ion time scale also include the electron's fluid equations. Ion-time-scale electron continuity equation can be obtained from $\left\langle \iiint (1^e)\, d^3u\right\rangle_i$, which yields,

$$\frac{\partial n^i}{\partial t} + \nabla\cdot(n^i\mathbf{V}_e{}^i) = 0 \qquad (5)$$

This equation (5) together with ions continuity equation can also be obtained from the divergence of Ampere's law. Ion-time-scale electron momentum equation can be obtained from $\left\langle \iiint m_e\mathbf{u}(1^e) d^3u\right\rangle_i$, which yields,

$$\frac{\partial}{\partial t}\left\langle \iiint m_e\mathbf{u} f_e{}^e d^3u\right\rangle_i + \nabla\cdot\left\langle \iiint m_e\mathbf{u}\,\mathbf{v}(\mathbf{u}) f_e{}^e d^3u\right\rangle_i$$
$$-(-e)\left\langle\left(\iiint f_e{}^e d^3u\right)\mathbf{E}^e + \left(\iiint \mathbf{v}(\mathbf{u}) f_e{}^e d^3u\right)\times\mathbf{B}^e\right\rangle_i = 0 \qquad (6)$$

where $m_e\mathbf{u}$ is the momentum of electrons with velocity equal to $\mathbf{v}(\mathbf{u}) = \mathbf{u}/[1+(u/c)^2]^{1/2}$. To formulate anomalous dissipation term, we need compare (6) with the momentum equation of ideal electron fluid,

$$\frac{\partial}{\partial t}\left(m_e n^i\mathbf{V}_e{}^i\right) + \nabla\cdot\left(\mathbf{1}p_e{}^i + m_e n^i\mathbf{V}_e{}^i\mathbf{V}_e{}^i\right)$$
$$-(-e)\left(n^i\mathbf{E}^i + n^i\mathbf{V}_e{}^i\times\mathbf{B}^i\right) = 0 \qquad (7)$$

where electron pressure is assumed to be isotropic and low-frequency modulations from high-frequency wave-wave interactions are ignored. If we define

$$\delta\mathbf{F}_e{}^i = -\nabla\cdot\left[\left\langle \iiint m_e\mathbf{u}\,\mathbf{v}(\mathbf{u}) f_e{}^e d^3u\right\rangle_i - \left(\mathbf{1}p_e{}^i + m_e n^i\mathbf{V}_e{}^i\mathbf{V}_e{}^i\right)\right]$$
$$+(-e)\left[\left\langle\left(\iiint f_e{}^e d^3u\right)\mathbf{E}^e\right\rangle_i - n^i\mathbf{E}^i\right]$$
$$+(-e)\left[\left\langle\left(\iiint \mathbf{v}(\mathbf{u}) f_e{}^e d^3u\right)\times\mathbf{B}^e\right\rangle_i - n^i\mathbf{V}_e{}^i\times\mathbf{B}^i\right] \qquad (8)$$

(6) can be rewritten as,

$$\frac{\partial}{\partial t}\left(m_e n^i\mathbf{U}_e{}^i\right) + \nabla\cdot\left(m_e n^i\mathbf{V}_e{}^i\mathbf{V}_e{}^i\right)$$
$$= -\nabla p_e{}^i + (-e)\left(n^i\mathbf{E}^i + n^i\mathbf{V}_e{}^i\times\mathbf{B}^i\right) + \delta\mathbf{F}_e{}^i \qquad (9)$$

where $m_e n^i\mathbf{U}_e{}^i$ is defined by $m_e n^i\mathbf{U}_e{}^i \equiv \left\langle \iiint m_e\mathbf{u} f_e{}^e d^3u\right\rangle_i$ which is the average momentum of relativistic electrons. (If most of the electrons are non-relativistic electrons, we could have $\mathbf{U}_e{}^i \approx \mathbf{V}_e{}^i$.) Since $m_e << m_i$, (9) can be approximately leads to

$$-\nabla p_e{}^i - e\left(n^i\mathbf{E}^i + n^i\mathbf{V}_e{}^i \times \mathbf{B}^i\right) + \delta\mathbf{F}_e{}^i \approx O(m_e/m_i) \to 0 \quad (11)$$

Therefore, we can obtain ion-time-scale Ohm's law,

$$\mathbf{E}^i \approx -\mathbf{V}_e{}^i \times \mathbf{B}^i + \frac{1}{en^i}\left[-\nabla p_e{}^i + \delta\mathbf{F}_e{}^i\right] \quad (11a)$$

or from (4), we have

$$\mathbf{E}^i \approx -\mathbf{V}_i{}^i \times \mathbf{B}^i + \frac{1}{en^i}\left[-\nabla p_e{}^i + \mathbf{J}^i \times \mathbf{B}^i + \delta\mathbf{F}_e{}^i\right] \quad (11b)$$

As we can see, in addition to the so-called Hall current effect in (11b), the anomalous dissipation term $\delta\mathbf{F}_e{}^i$ may play an essential role on the current dissipation during fast energy release processes, such as magnetic reconnection, in space plasmas. Since electromagnetic energy dissipation can be evaluated by $\mathbf{J}\cdot\mathbf{E}$, the Hall current term has no direct contribution to the electromagnetic energy dissipation. The pressure gradient term in (11b) can build up an electrostatic potential, which may not associate with the dissipation of energy stored in the magnetic field. Therefore, $\delta\mathbf{F}_e{}^i$ may be the only one that can provide anomalous resistivity in collisionless plasma. Physical processes associated with $\delta\mathbf{F}_e{}^i$ can be seen from the three terms on the left-hand side of equation (8). The first term represents a viscous force due to anisotropic pressure in electron fluid. The second term can be identified as the classical ponderomotive force in electrostatic plasma (e.g., Nicholson, 1983). The third term may be considered as an electromagnetic version of ponderomotive force in the collisionless plasma.

Ion-time-scale electron energy equation can be obtained from $\left\langle \iiint m_e c^2(\gamma(u)-1)\,(1^e)d^3u \right\rangle_i$, which yields,

$$\frac{\partial}{\partial t}\left\langle \iiint m_e c^2(\gamma(u)-1) f_e{}^e d^3u \right\rangle_i$$
$$+\nabla\cdot\left\langle \iiint m_e c^2(\gamma(u)-1)\,\mathbf{v}(\mathbf{u})\, f_e{}^e d^3u \right\rangle_i$$
$$-(-e)\left\langle\left(\iiint \mathbf{v}(\mathbf{u})\, f_e{}^e d^3u\right)\cdot\mathbf{E}^e\right\rangle_i = 0 \quad (12)$$

where $m_e c^2(\gamma(u)-1)$ is the kinetic energy of electrons with velocity equal to $\mathbf{v}(\mathbf{u}) = \mathbf{u}/[1+(u/c)^2]^{1/2}$. For $u << c$, we should use Taylor's expansion in evaluating $m_e c^2(\gamma(u)-1)$ to reduce numerical error in the calculation. To formulate anomalous dissipation term, we need compare (12) with the energy equation of ideal electron fluid,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}m_e n^i \mathbf{V}_e{}^i\cdot\mathbf{V}_e{}^i + \frac{3}{2}p_e{}^i\right)$$
$$+\nabla\cdot\left[\left(\frac{1}{2}m_e n^i \mathbf{V}_e{}^i\cdot\mathbf{V}_e{}^i + \frac{5}{2}p_e{}^i\right)\mathbf{V}_e{}^i\right] - (-e)\left(n^i\mathbf{V}_e{}^i\cdot\mathbf{E}^i\right) = 0 \quad (13a)$$

If we use $K_e{}^i$ to denote the total kinetic energy of electrons, which is $(1/2)m_e n^i\mathbf{V}_e{}^i\cdot\mathbf{V}_e{}^i + (3/2)p_e{}^i$ for the ideal electron fluid, (13a) can be rewritten as

$$\frac{\partial}{\partial t}\left(K_e{}^i\right) + \nabla\cdot\left[\left(K_e{}^i + p_e{}^i\right)\mathbf{V}_e{}^i\right] - (-e)\left(n^i\mathbf{V}_e{}^i\cdot\mathbf{E}^i\right) = 0 \quad (13b)$$

and the electron pressure can be obtained from $K_e{}^i$, i.e,

$$p_e{}^i = \frac{2}{3}\left(K_e{}^i - \frac{1}{2}m_e n^i \mathbf{V}_e{}^i\cdot\mathbf{V}_e{}^i\right) \quad (14)$$

If we define

$$\delta\varepsilon_e{}^i = -\nabla\cdot\left[\left\langle \iiint m_e c^2[\mathbf{u}-\mathbf{v}(\mathbf{u})] f_e{}^e d^3u \right\rangle_i - \left(K_e{}^i + p_e{}^i\right)\mathbf{V}_e{}^i\right]$$
$$-e\left[\left\langle\left(\iiint \mathbf{v}(\mathbf{u}) f_e{}^e d^3u\right)\cdot\mathbf{E}^e\right\rangle_i - n^i\mathbf{V}_e{}^i\cdot\mathbf{E}^i\right] \quad (15)$$

(12) can be rewritten as,

$$\frac{\partial}{\partial t}\left(K_e{}^i\right) = -\nabla\cdot\left[\left(K_e{}^i + p_e{}^i\right)\mathbf{V}_e{}^i\right] - e\left(n^i\mathbf{V}_e{}^i\cdot\mathbf{E}^i\right) + \delta\varepsilon_e{}^i \quad (16)$$

where the total kinetic energy of relativistic electron fluid in the ion time scale is defined by $K_e{}^i \equiv \left\langle \iiint m_e c^2(\gamma(u)-1) f_e{}^e d^3u \right\rangle_i$.

But it will be solved from (16) in our ion-time-scale simulation model.

From equations (1^i)~(16), we can obtain governing equations for ion-time-scale simulation, which are summarized in Table 2. Algorithm for solving this set of equations will be discussed in section III.

3. Basic equations for quasi-MHD and MHD time scale plasma

Basic equations for MHD time scale plasma consist of Maxwell's equations and one-fluid plasma equations. Maxwell's equations in MHD time scale can be obtained from time average of (2^i), which yields

$$\nabla\cdot\mathbf{E}^{MHD} = \rho_c{}^{MHD}/\varepsilon_\circ$$
$$\nabla\cdot\mathbf{B}^{MHD} = 0$$
$$\frac{\partial\mathbf{B}^{MHD}}{\partial t} = -\nabla\times\mathbf{E}^{MHD} \quad (2^{MHD})$$
$$\nabla\times\mathbf{B}^{MHD} \approx \mu_\circ\mathbf{J}^{MHD}$$

where superscript *MHD* denotes MHD-time-scale variables. Since the MHD-time-scale charge density $\rho_c{}^{MHD}$ is very small, we shall not use the Poisson equation in (2^{MHD}) to determine MHD-time-scale electric field $\mathbf{E}^{MHD}$. One fluid continuity equation can be obtained from $\left\langle m_e(5) + \iiint m_i\,(1^i)d^3v\right\rangle_{MHD}$, which yields,

$$\frac{\partial}{\partial t}\left(\rho^{MHD}\right) + \nabla\cdot\left(\rho^{MHD}\mathbf{V}^{MHD}\right) = 0 \quad (17)$$

where notation $\left\langle A^i\right\rangle_{MHD}$ denotes $\int_0^{100\tau_i} A^i(t)dt \big/ 100\tau_i$, and the following definitions of one-fluid mass density ρ^{MHD} and flow velocity $\mathbf{V}^{MHD}$ have been used in (17).

$$\rho^{MHD} \equiv (m_i + m_e)\left\langle n^i\right\rangle_{MHD}$$
$$\mathbf{V}^{MHD} \equiv \left[m_i\left\langle \iiint \mathbf{v} f_i{}^i d^3v\right\rangle_{MHD} + m_e\left\langle n^i V_e{}^i\right\rangle_{MHD}\right]\Big/\rho^{MHD}$$

The ion number density and flow velocity in the MHD time scale can then be determined directly from these one-fluid variables

$$n_i{}^{MHD} \approx n^{MHD} = \rho^{MHD}/(m_i + m_e)$$
$$\mathbf{V}_i{}^{MHD} = \mathbf{V}^{MHD} + \left[m_e/(m_i + m_e)\right]\left(\mathbf{J}^{MHD}/e\,n^{MHD}\right) \quad (18)$$

They will serve as the boundary and initial condition for ion-time-scale simulation. Likewise, we have

$$\mathbf{V}_e{}^{MHD} = \mathbf{V}^{MHD} - \left[m_i/(m_i + m_e)\right]\left(\mathbf{J}^{MHD}/e\,n^{MHD}\right) \quad (19a)$$

Since $m_e << m_i$, (19a) is approximately equal to,

$$\mathbf{V}_e{}^{MHD} \approx \mathbf{V}^{MHD} - \left(\mathbf{J}^{MHD}/e\,n^{MHD}\right) \quad (19b)$$

One fluid momentum equation can be obtained from $\left\langle (9) + \iiint m_i \mathbf{v}\,(1^i) d^3 v \right\rangle_{MHD}$, which yields,

$$\frac{\partial}{\partial t}\left[\left\langle m_e n^i \mathbf{U}_e{}^i + m_i \iiint \mathbf{v} f_i{}^i d^3 v \right\rangle_{MHD}\right] + \nabla \cdot \left[\left\langle \mathbf{1} p_e{}^i + m_e n^i \mathbf{V}_e{}^i \mathbf{V}_e{}^i + m_i \iiint \mathbf{v}\mathbf{v} f_i{}^i d^3 v \right\rangle_{MHD}\right] = \left\langle \mathbf{J}^i \times \mathbf{B}^i + \delta \mathbf{F}_e{}^i \right\rangle_{MHD} \quad (20)$$

Following the similar procedure as described in the last section, we can define an anomalous dissipation force

$$\delta \mathbf{F}^{MHD} = \left\langle \delta \mathbf{F}_e{}^i \right\rangle_{MHD} + \left[\left\langle \mathbf{J}^i \times \mathbf{B}^i \right\rangle_{MHD} - \mathbf{J}^{MHD} \times \mathbf{B}^{MHD}\right] - \nabla \cdot \left[\left\langle \mathbf{1} p_e{}^i + m_e n^i \mathbf{V}_e{}^i \mathbf{V}_e{}^i + m_i \iiint \mathbf{v}\mathbf{v} f_i{}^i d^3 v \right\rangle_{MHD} - \left(\mathbf{1} p^{MHD} + \rho^{MHD} \mathbf{V}^{MHD} \mathbf{V}^{MHD}\right)\right] \quad (21)$$

so that (20) can be rewritten in the conventional form

$$\frac{\partial}{\partial t}\left(\rho^{MHD} \mathbf{V}^{MHD}\right) + \nabla \cdot \left(\rho^{MHD} \mathbf{V}^{MHD} \mathbf{V}^{MHD}\right) = -\nabla p^{MHD} + \mathbf{J}^{MHD} \times \mathbf{B}^{MHD} + \delta \mathbf{F}^{MHD} \quad (22)$$

where for simplicity, we have assume that

$$\left\langle m_e n^i \mathbf{U}_e{}^i + m_i \iiint \mathbf{v} f_i{}^i d^3 v \right\rangle_{MHD} \approx \left\langle m_e n^i \mathbf{V}_e{}^i + m_i \iiint \mathbf{v} f_i{}^i d^3 v \right\rangle_{MHD} = \rho^{MHD} \mathbf{V}^{MHD}$$

Errors rising from this assumption are negligible, because $m_e << m_i$. One fluid energy equation can be obtained from $\left\langle (16) + \iiint m_i \frac{\mathbf{v} \cdot \mathbf{v}}{2} (1^i) d^3 v \right\rangle_{MHD}$, which yields,

$$\frac{\partial}{\partial t}\left[\left\langle K_e{}^i + \iiint m_i \frac{\mathbf{v} \cdot \mathbf{v}}{2} f_i{}^i d^3 v \right\rangle_{MHD}\right] = -\nabla \cdot \left[\left\langle \left(K_e{}^i + p_e{}^i\right) \mathbf{V}_e{}^i + \iiint m_i \frac{\mathbf{v} \cdot \mathbf{v}}{2} \mathbf{v} f_i{}^i d^3 v \right\rangle_{MHD}\right] + \left\langle \mathbf{J}^i \cdot \mathbf{E}^i + \delta \varepsilon_e{}^i \right\rangle_{MHD} \quad (23)$$

Again, we can define an anomalous dissipation term

$$\delta \varepsilon^{MHD} = \left\langle \delta \varepsilon_e{}^i \right\rangle_{MHD} + \left[\left\langle \mathbf{J}^i \cdot \mathbf{E}^i \right\rangle_{MHD} - \mathbf{J}^{MHD} \cdot \mathbf{E}^{MHD}\right] - \nabla \cdot \left[\left\langle \left(K_e{}^i + p_e{}^i\right) \mathbf{V}_e{}^i + \iiint m_i \frac{\mathbf{v} \cdot \mathbf{v}}{2} \mathbf{v} f_i{}^i d^3 v \right\rangle_{MHD} - \left(K^{MHD} + p^{MHD}\right) \mathbf{V}^{MHD}\right] \quad (24)$$

so that (23) can be rewritten in the conventional form

$$\frac{\partial}{\partial t}\left(K^{MHD}\right) = -\nabla \cdot \left[\left(K^{MHD} + p^{MHD}\right) \mathbf{V}^{MHD}\right] + \mathbf{J}^{MHD} \cdot \mathbf{E}^{MHD} + \delta \varepsilon^{MHD} \quad (25)$$

where by definition, we have

$$K^{MHD} = \frac{3}{2} p_e{}^{MHD} + \frac{1}{2} m_e n^{MHD} \mathbf{V}_e{}^{MHD} \cdot \mathbf{V}_e{}^{MHD} + \frac{3}{2} p_i{}^{MHD} + \frac{1}{2} m_i n^{MHD} \mathbf{V}_i{}^{MHD} \cdot \mathbf{V}_i{}^{MHD} \quad (26)$$

and

$$p^{MHD} = p_e{}^{MHD} + p_i{}^{MHD} \quad (27)$$

Substituting (18), (19), and (27) into (26), yields,

$$K^{MHD} = \frac{3}{2} p^{MHD} + \frac{1}{2} \rho^{MHD} \left(V^{MHD}\right)^2 + \frac{1}{2} \frac{m_i m_e}{m_i + m_e} \frac{\left(J^{MHD}\right)^2}{n^{MHD} e^2} \quad (28)$$

For $m_e << m_i$, (28) yields,

$$p^{MHD} \approx \frac{2}{3}\left[K^{MHD} - \frac{1}{2} \rho^{MHD} \left(V^{MHD}\right)^2\right] \quad (29)$$

Namely, we can solve K^{MHD} from (25) and determine p^{MHD} from (29).

To obtain MHD-time-scale electric field, we need to derive the Ohm's law in MHD time scale. We first take time average of ion-time-scale Ohm's law, $\langle (11) \rangle_{MHD}$, which yields,

$$-\nabla p_e{}^{MHD} - e\left[\left\langle n^i \mathbf{E}^i + n^i \mathbf{V}_e{}^i \times \mathbf{B}^i \right\rangle_{MHD}\right] + \left\langle \delta \mathbf{F}_e{}^i \right\rangle_{MHD} \approx 0 \quad (30)$$

Following the similar procedure as described in the last section, we can define an anomalous dissipation force

$$\delta \mathbf{F}_e{}^{MHD} = \left\langle \delta \mathbf{F}_e{}^i \right\rangle_{MHD} + (-e)\left[\left\langle n^i \mathbf{E}^i + n^i \mathbf{V}_e{}^i \times \mathbf{B}^i \right\rangle_{MHD} - n^{MHD}\left(\mathbf{E}^{MHD} + \mathbf{V}_e{}^{MHD} \times \mathbf{B}^{MHD}\right)\right] \quad (31)$$

so that (30) can be rewritten in a more conventional form

$$-\nabla p_e{}^{MHD} - e n^{MHD}\left(\mathbf{E}^{MHD} + \mathbf{V}_e{}^{MHD} \times \mathbf{B}^{MHD}\right) + \delta \mathbf{F}_e{}^{MHD} \approx 0 \quad (32)$$

Using (19b), we rewrite (32) in the following format

$$\mathbf{E}^{MHD} \approx -\mathbf{V}^{MHD} \times \mathbf{B}^{MHD} + \frac{1}{e n^{MHD}}\left[-\nabla p_e{}^{MHD} + \mathbf{J}^{MHD} \times \mathbf{B}^{MHD} + \delta \mathbf{F}_e{}^{MHD}\right] \quad (33)$$

In order to determine $p_e{}^{MHD}$ in the Ohm's law (33), we need to evaluate electron energy equation in the MHD time scale. From $\langle (16) \rangle_{MHD}$, we have

$$\frac{\partial}{\partial t}\left(K_e{}^{MHD}\right) = -\nabla \cdot \left[\left\langle \left(K_e{}^i + p_e{}^i\right) \mathbf{V}_e{}^i \right\rangle_{MHD}\right] - e\left[\left\langle n^i \mathbf{V}_e{}^i \cdot \mathbf{E}^i \right\rangle_{MHD}\right] + \left\langle \delta \varepsilon_e{}^i \right\rangle_{MHD} \quad (34)$$

Following the similar procedure as described in the last section, we can define an anomalous dissipation term

$$\delta \varepsilon_e{}^{MHD} = \left\langle \delta \varepsilon_e{}^i \right\rangle_{MHD} - e\left[\left\langle n^i \mathbf{V}_e{}^i \cdot \mathbf{E}^i \right\rangle_{MHD} - n^{MHD} \mathbf{V}_e{}^{MHD} \cdot \mathbf{E}^{MHD}\right] - \nabla \cdot \left[\left\langle \left(K_e{}^i + p_e{}^i\right) \mathbf{V}_e{}^i \right\rangle_{MHD} - \left(K_e{}^{MHD} + p_e{}^{MHD}\right) \mathbf{V}_e{}^{MHD}\right] \quad (35)$$

so that (34) can be rewritten in the conventional form

$$\frac{\partial}{\partial t}\left(K_e^{MHD}\right) = -\nabla \cdot \left[\left(K_e^{MHD} + p_e^{MHD}\right)\mathbf{V}_e^{MHD}\right] - e\left(n^{MHD}\mathbf{V}_e^{MHD} \cdot \mathbf{E}^{MHD}\right) + \delta\varepsilon_e^{MHD} \quad (36)$$

From definition, we have

$$K_e^{MHD} = \frac{3}{2} p_e^{MHD} + \frac{1}{2} m_e n^{MHD} \mathbf{V}_e^{MHD} \cdot \mathbf{V}_e^{MHD} \quad (37)$$

Therefore, we can solve K_e^{MHD} from (36) and determine p_e^{MHD} from (37).

From equations (2^{MHD}) and (17)~(37), we can obtain governing equations for MHD-time-scale simulation, which are summarized in Table 3. Algorithm for solving this set of equations will be discussed in the next section.

III. Algorithm for Cross-Scale Simulation

Coupling effects between different time scales are included in each set of governing equations. In our model, large-scale low-frequency simulations will provide initial and boundary conditions for small-scale simulations. On the other hand, anomalous dissipations due to wave-wave and wave-particle interactions can be obtained from small-scale high-frequency simulations and will be feedback into large-scale low-frequency simulation. Global distribution of anomalous dissipation terms can be obtained from a few sampled points where small-scale high-frequency simulations were carried out. Since small-scale simulations need to be carried out at sampled points near strong gradient region, boundary conditions of small-scale simulations are generally non-periodic. Due to the discrete nature in the traditional particle code simulation, it will be rather difficult to handle particle flux at boundaries with this type of non-periodic boundary conditions. Therefore, we suggest solving electrons' and ions' Vlasov equation directly in the phase space. The derivatives and integrations in the velocity and space domain can be obtained from cubic-spline method. Higher order predict-and-correct scheme is recommended to solve the set of first-order time-derivative differential equations in each time intervals. An adaptive scheme is recommended to determine where the small-scale simulations need to be carried out near strong gradient region. The coupling variables: $\delta\mathbf{F}_e^{\,i}$, $\delta\varepsilon_e^{\,i}$, $\delta\mathbf{F}_e^{MHD}$, $\delta\mathbf{F}^{MHD}$, $\delta\varepsilon_e^{MHD}$, and $\delta\varepsilon^{MHD}$, can be evaluated directly at these sample points, where small-scale high-frequency simulations are carried out. Since the anomalous dissipations are usually small and can be ignored in small gradient regions, the global distribution of these coupling variables can then be obtained from an interpolated fitting function, such as cubic spline, over these sample points and with uniform and zero boundary conditions at zero gradient region.

IV. Summary

A cross-scale simulation scheme has been proposed in this study. Governing equations for electron-scale, ion-scale, and macro-scale simulations are summarized in Tables 1, 2, and 3, respectively. Under the assumption that small-scale high-frequency wave-wave and wave-particle interactions can lead to a low-frequency impact on large-scale phenomena, anomalous dissipations are evaluated by taking differences between (A) and (B), where (A) is a time-averaging of high-frequency momentum or energy equation, and (B) is the conventional formulation of low-frequency momentum or energy equation of an isentropic plasma. As a result, anomalous dissipations are fully included in the coupling variables $\delta\mathbf{F}_e^{\,i}$, $\delta\varepsilon_e^{\,i}$, $\delta\mathbf{F}_e^{MHD}$, $\delta\mathbf{F}^{MHD}$, $\delta\varepsilon_e^{MHD}$, and $\delta\varepsilon^{MHD}$ in Table 2 and Table 3. A brief discussion on the physical processes associated with $\delta\mathbf{F}_e^{\,i}$ has been given in section II. Similar physical processes can be expected for other coupling variables. The values of these coupling variables can be obtained directly from the sampled points, where small-scale simulations are carried out. A global distribution of these variables can then be obtained from an interpolated fitting function over these sampled points. As a result, the cross-scale simulation scheme proposed in this study can model plasma processes from MHD-scale to electron-scale. This simulation scheme requires much less memory space and computing time in comparison with a simulation carried out by a single high-frequency simulation code.

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Table 1. Governing equation for electron-time-scale simulation.

Governing Equation	Output
$\frac{\partial f_e{}^e}{\partial t} = -\mathbf{v}(\mathbf{u})\cdot\frac{\partial f_e{}^e}{\partial \mathbf{x}} - \frac{-e}{m_e}(\mathbf{E}^e+\mathbf{v}(\mathbf{u})\times\mathbf{B}^e)\cdot\frac{\partial f_e{}^e}{\partial \mathbf{u}}$ where $\mathbf{v}(\mathbf{u}) = \mathbf{u}/[1+(u/c)^2]^{1/2}$	$f_e{}^e$
$\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{B}^e}{\partial t}\right) = c^2\left(\nabla^2\mathbf{B}^e+\mu_\circ\nabla\times\mathbf{J}^e\right)$	$\frac{\partial\mathbf{B}^e}{\partial t}$
$\frac{\partial}{\partial t}\left(\mathbf{B}^e\right) = \frac{\partial \mathbf{B}^e}{\partial t}$	$\mathbf{B}^e$
$\rho_c{}^e = e(n_i{}^i - \iiint f_e{}^e d^3u)$ $\mathbf{J}^e = e[n_i{}^i\mathbf{V}_i{}^i - \iiint \mathbf{v}(\mathbf{u}) f_e{}^e d^3u]$	$\rho_c{}^e$ $\mathbf{J}^e$
$\nabla^2\phi^e = -\rho_c{}^e/\varepsilon_\circ$ $\mathbf{E}^{e,ES} = -\nabla\phi^e$ $\nabla^2\mathbf{a}^e = \partial\mathbf{B}^e/\partial t$ $\mathbf{E}^{e,EM} = \nabla\times\mathbf{a}^e$ $\mathbf{E}^e = \mathbf{E}^{e,ES}+\mathbf{E}^{e,EM}$	ϕ^e $\mathbf{E}^{e,ES}$ $\mathbf{a}^e$ $\mathbf{E}^{e,EM}$ $\mathbf{E}^e$

Table 2. Governing equation for ion-time-scale simulation.

Governing Equation	Output
$\frac{\partial f_i{}^i}{\partial t} = -\mathbf{v}\cdot\frac{\partial f_i{}^i}{\partial \mathbf{x}} - \frac{+e}{m_i}(\mathbf{E}^i+\mathbf{v}\times\mathbf{B}^i)\cdot\frac{\partial f_i{}^i}{\partial \mathbf{v}}$	$f_i{}^i$
$\frac{\partial \mathbf{B}^i}{\partial t} = -\nabla\times\mathbf{E}^i$	$\mathbf{B}^i$
$\frac{\partial K_e{}^i}{\partial t} = -\nabla\cdot\left[\left(K_e{}^i+p_e{}^i\right)\mathbf{V}_e{}^i\right] - e\left(n^i\mathbf{V}_e{}^i\cdot\mathbf{E}^i\right)+\delta\varepsilon_e{}^i$	$K_e{}^i$
$\mathbf{J}^i = \nabla\times\mathbf{B}^i/\mu_\circ$ $n_i{}^i \equiv \iiint f_i{}^i d^3v$ $n_e{}^i \approx n_i{}^i$ $n^i \approx n_i{}^i$ $\mathbf{V}_i{}^i \equiv \iiint \mathbf{v} f_i{}^i d^3v \big/ \iiint f_i{}^i d^3v$ $\mathbf{V}_e{}^i = \mathbf{V}_i{}^i - \left(\mathbf{J}^i/en^i\right)$	$\mathbf{J}^i$ $n_i{}^i$ $n_e{}^i$ n^i $\mathbf{V}_i{}^i$ $\mathbf{V}_e{}^i$
$p_e{}^i = \frac{2}{3}\left(K_e{}^i - \frac{1}{2}m_e n^i\mathbf{V}_e{}^i\cdot\mathbf{V}_e{}^i\right)$	$p_e{}^i$
$\mathbf{E}^i \approx -\mathbf{V}_i{}^i\times\mathbf{B}^i + \frac{1}{en^i}\left[-\nabla p_e{}^i + \mathbf{J}^i\times\mathbf{B}^i + \delta\mathbf{F}_e{}^i\right]$	$\mathbf{E}^i$
$\delta\mathbf{F}_e{}^i = -\nabla\cdot\left[\left\langle \iiint m_e\mathbf{u}\,\mathbf{v}(\mathbf{u}) f_e{}^e d^3u\right\rangle_i - \left(\mathbf{1}p_e{}^i + m_e n^i\mathbf{V}_e{}^i\mathbf{V}_e{}^i\right)\right]$ $+(-e)\left[\left\langle\left(\iiint f_e{}^e d^3u\right)\mathbf{E}^e\right\rangle_i - n^i\mathbf{E}^i\right]$ $+(-e)\left[\left\langle\left(\iiint \mathbf{v}(\mathbf{u}) f_e{}^e d^3u\right)\times\mathbf{B}^e\right\rangle_i - n^i\mathbf{V}_e{}^i\times\mathbf{B}^i\right]$	$\delta\mathbf{F}_e{}^i$
$\delta\varepsilon_e{}^i = -\nabla\cdot\left[\left\langle\iiint m_e c^2[\mathbf{u}-\mathbf{v}(\mathbf{u})] f_e{}^e d^3u\right\rangle_i - \left(K_e{}^i+p_e{}^i\right)\mathbf{V}_e{}^i\right]$ $-e\left[\left\langle\left(\iiint \mathbf{v}(\mathbf{u}) f_e{}^e d^3u\right)\cdot\mathbf{E}^e\right\rangle_i - n^i\mathbf{V}_e{}^i\cdot\mathbf{E}^i\right]$	$\delta\varepsilon_e{}^i$

where $\mathbf{v}(\mathbf{u}) = \mathbf{u}/[1+(u/c)^2]^{1/2}$

Table 3. Governing equation for MHD-time-scale simulation.

Governing Equation	Output
$\frac{\partial \mathbf{B}^{MHD}}{\partial t} = -\nabla\times\mathbf{E}^{MHD}$	$\mathbf{B}^{MHD}$
$\frac{\partial \rho^{MHD}}{\partial t} = -\nabla\cdot\left(\rho^{MHD}\mathbf{V}^{MHD}\right)$	ρ^{MHD}
$\frac{\partial(\rho\mathbf{V})^{MHD}}{\partial t} = -\nabla\cdot\left(\rho^{MHD}\mathbf{V}^{MHD}\mathbf{V}^{MHD}\right)$ $-\nabla p^{MHD} + \mathbf{J}^{MHD}\times\mathbf{B}^{MHD} + \delta\mathbf{F}^{MHD}$	$(\rho\mathbf{V})^{MHD}$
$\frac{\partial K^{MHD}}{\partial t} = -\nabla\cdot\left[\left(K^{MHD}+p^{MHD}\right)\mathbf{V}^{MHD}\right]$ $+\mathbf{J}^{MHD}\cdot\mathbf{E}^{MHD} + \delta\varepsilon^{MHD}$	K^{MHD}
$\frac{\partial K_e{}^{MHD}}{\partial t} = -\nabla\cdot\left[\left(K_e{}^{MHD}+p_e{}^{MHD}\right)\mathbf{V}_e{}^{MHD}\right]$ $-e\left(n^{MHD}\mathbf{V}_e{}^{MHD}\cdot\mathbf{E}^{MHD}\right)+\delta\varepsilon_e{}^{MHD}$	$K_e{}^{MHD}$
$\mathbf{J}^{MHD} \approx \nabla\times\mathbf{B}^{MHD}/\mu_\circ$ $n^{MHD} = \rho^{MHD}/(m_i+m_e)$ $n_i{}^{MHD} \approx n^{MHD}$ $n_e{}^{MHD} \approx n^{MHD}$ $\mathbf{V}_i{}^{MHD} = \mathbf{V}^{MHD} + [m_e/(m_i+m_e)](\mathbf{J}^{MHD}/e\,n^{MHD})$ $\mathbf{V}_e{}^{MHD} = \mathbf{V}^{MHD} - [m_i/(m_i+m_e)](\mathbf{J}^{MHD}/e\,n^{MHD})$	$\mathbf{J}^{MHD}$ n^{MHD} $n_i{}^{MHD}$ $n_e{}^{MHD}$ $\mathbf{V}_i{}^{MHD}$ $\mathbf{V}_e{}^{MHD}$
$p^{MHD} \approx \frac{2}{3}\left[K^{MHD} - \frac{1}{2}\rho^{MHD}\left(V^{MHD}\right)^2\right]$ $p_e{}^{MHD} = \frac{2}{3}\left[K_e{}^{MHD} - \frac{1}{2}m_e n^{MHD}\left(V_e{}^{MHD}\right)^2\right]$ $p_i{}^{MHD} = p^{MHD} - p_e{}^{MHD}$	p^{MHD} $p_e{}^{MHD}$ $p_i{}^{MHD}$
$\mathbf{E}^{MHD} \approx -\mathbf{V}^{MHD}\times\mathbf{B}^{MHD}$ $+\frac{1}{en^{MHD}}\left[-\nabla p_e{}^{MHD} + \mathbf{J}^{MHD}\times\mathbf{B}^{MHD} + \delta\mathbf{F}_e{}^{MHD}\right]$	$\mathbf{E}^{MHD}$
$\delta\mathbf{F}^{MHD} = \left\langle\delta\mathbf{F}_e{}^i\right\rangle_{MHD} + \left[\left\langle\mathbf{J}^i\times\mathbf{B}^i\right\rangle_{MHD} - \mathbf{J}^{MHD}\times\mathbf{B}^{MHD}\right]$ $-\nabla\cdot\left[\left\langle\mathbf{1}p_e{}^i + m_e n^i\mathbf{V}_e{}^i\mathbf{V}_e{}^i + m_i\iiint \mathbf{v}\mathbf{v} f_i{}^i d^3v\right\rangle_{MHD}\right.$ $\left.-\left(\mathbf{1}p^{MHD} + \rho^{MHD}\mathbf{V}^{MHD}\mathbf{V}^{MHD}\right)\right]$	$\delta\mathbf{F}^{MHD}$
$\delta\varepsilon^{MHD} = \left\langle\delta\varepsilon_e{}^i\right\rangle_{MHD} + \left[\left\langle\mathbf{J}^i\cdot\mathbf{E}^i\right\rangle_{MHD} - \mathbf{J}^{MHD}\cdot\mathbf{E}^{MHD}\right]$ $-\nabla\cdot\left[\left\langle\left(K_e{}^i+p_e{}^i\right)\mathbf{V}_e{}^i + \iiint m_i\frac{\mathbf{v}\cdot\mathbf{v}}{2}\mathbf{v} f_i{}^i d^3v\right\rangle_{MHD}\right.$ $\left.-\left(K^{MHD}+p^{MHD}\right)\mathbf{V}^{MHD}\right]$	$\delta\varepsilon^{MHD}$
$\delta\mathbf{F}_e{}^{MHD} = \left\langle\delta\mathbf{F}_e{}^i\right\rangle_{MHD} + (-e)\left[\left\langle n^i\mathbf{E}^i + n^i\mathbf{V}_e{}^i\times\mathbf{B}^i\right\rangle_{MHD}\right.$ $\left.-n^{MHD}\left(\mathbf{E}^{MHD}+\mathbf{V}_e{}^{MHD}\times\mathbf{B}^{MHD}\right)\right]$	$\delta\mathbf{F}_e{}^{MHD}$
$\delta\varepsilon_e{}^{MHD} = \left\langle\delta\varepsilon_e{}^i\right\rangle_{MHD}$ $-\nabla\cdot\left[\left\langle\left(K_e{}^i+p_e{}^i\right)\mathbf{V}_e{}^i\right\rangle_{MHD} - \left(K_e{}^{MHD}+p_e{}^{MHD}\right)\mathbf{V}_e{}^{MHD}\right]$ $-e\left[\left\langle n^i\mathbf{V}_e{}^i\cdot\mathbf{E}^i\right\rangle_{MHD} - n^{MHD}\mathbf{V}_e{}^{MHD}\cdot\mathbf{E}^{MHD}\right]$	$\delta\varepsilon_e{}^{MHD}$